1991

Quantum mechanics in the real Pauli algebra.

Jiansu. Wei
University of Windsor

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

QUANTUM MECHANICS IN THE REAL PAULI ALGEBRA

by

Jiansu WEI

A Thesis
Submitted to the Faculty of Graduate Studies and Research
through the Department of Physics in
Partial Fulfillment of the Requirements for the
Degree of Master of Science at the
University of Windsor

Windsor, Ontario, CANADA

1991

ISBN 0-315-69881-0

ABSTRACT

In this thesis, real quantum mechanics (QM) is constructed over the field $\mathcal{F}_{\mathcal{P}_0 \oplus \mathcal{P}_3}$ in the Pauli algebra, where the trivector $i = \sigma_1 \sigma_2 \sigma_3$ arises naturally as a product of the unit basis vectors in 3-dimensional Euclidean space. In this scheme, i is the volume element of the algebra with $i^2 = -1$ and has geometrical meaning. Components of the state vector in Hilbert space take the form of a real element in the field $\mathcal{F}_{\mathcal{P}_0 \oplus \mathcal{P}_3}$ and the hermitean conjugate of the state vector is just its transpose, so that all the observables, represented by hermitean operators, are symmetric; i itself is an anti-hermitean operator. In this construction, the time reversal operator T is a combination of the transpose " + " and the unitary operator which changes the sign of t. Since T is an anti-automorphism the interpretation of its action must be carefully analyzed in terms of the state vectors and their inner products. Furthermore, the parity transformation is not represented by the spatial inversion $\tilde{\Pi}$ (an antilinear, anti-unitary transformation). Instead, the spatial reversal $\overline{\Pi}$ (an anti-automorphic, linear transformation) is proposed as a parity operator. It is unitary as well as hermitean, and it commutes with the Hamiltonian, H, of a system, thus, it is a constant of motion. This thesis mainly deals with non-relativistic QM for spinless particles.

DEDICATION

This thesis is dedicated to

My lovely wife Qiong LI

ACKNOWLEDGEMENTS

I would like to express my thanks and appreciation to Dr. W. E. Baylis for guiding me throughout the work leading to this thesis. Sincere thanks are also due to fellow students George Jones and David E Kosokowsky for their valuable discussions and help. Finally, the support of the Natural Sciences and Engineering Research Council of Canada and Department of Physics at the University of Windsor is gratefully acknowledged.

TABLE OF CONTENTS

CHAPTER 1. INTRODUCTION

"When we met the equation

$$i^2 = -1 \tag{1.1}$$

for the first time in our lives, we were probably worried because we could not conceive of a God-made object with this property"[1]. The imaginary number "i" had been rejected by the mathematicians for more than one hundred years after it was first introduced (Cardan, 1545), and even when it was accepted, G. Leibniz (1646-1716) marveled that it was "a fine and wonderful refuge of the divine spirit, almost an amphibian between being and non-being"[2].

From the mathematical view point, the imaginary number "i" originates through the solutions to the algebra equation

$$x^2 + 1 = 0. \tag{1.2}$$

In a geometrical sense "i" rotates vectors in the Argand (1805) plane by $\frac{1}{2}\pi$, while "i^2" rotates them by π, i.e., multiplies them by "-1". The latter provides an immediate realization of eq.(1.1) as a compound rotation. Hamilton (1843) expanded the concept of "i" to the quaternions

$$q = q_0 + q_1 i + q_2 j + q_3 k \tag{1.3}$$

where

$$i^2 = j^2 = k^2 = -1$$

$$ij = k, \quad ji = -k \tag{1.4}$$

with cyclic permutations $i \to j \to k \to i$. This provides a beautiful framework for rotations in three dimensions. The rotation from a vector $\vec{r}_1$ to $\vec{r}_2$, for example, takes the form

$$\vec{r}_2 = q\vec{r}_1 q^{-1}. \tag{1.5}$$

The Clifford (1878) algebras $C_{p,q}$ (a generalization of quaternions to n-dimensional space) provide another way to realize the square root of minus one. In $C_{p,q}$ it enters as the volume element

$$\eta = e_1 e_2 \cdots e_{p+q} \tag{1.6}$$

whenever $p - q = 2 \bmod 4$ or $p - q = 3 \bmod 4$.

The wonder increases when we meet "i" in physics. Physics is a human discipline designed to describe the real physical world. All the observations and measurements from classical mechanics to quantum physics are referred to real quantities. In physics, however, the use of "i" in a mathematical description of the physical world ranges from mechanical waves, electronics and electromagnetic theory to quantum theory. In classical physics, the description is just a computational aid, i.e., the physics is conceptually in terms of real numbers. The complex numbers can be separated into two real parts, and the imaginary number "i" can be eliminated without destroying the relevance to the physical system. In quantum mechanics on the other hand, things are very different. There is no way to eliminate "i" without destroying the meaning of quantum theory.

In quantum mechanics, there are at least three different ways that the "i" comes into play[3].First, "i" is associated with the fundamental equations of matrix mechanics and of wave mechanics

$$[P_x, X] = -i\hbar \tag{1.7}$$

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi. \qquad (1.8)$$

The corresponding classical canonical variables x and p_x satisfy the Poisson bracket $\{x, p_x\} = 1$, and "i" plays no role at all. It is to be emphasized that the physical meanings of these equations would be totally destroyed if one tried to eliminate the "i" by separating eqs.(1.7) and (1.8) in terms of real and imaginary parts[4].

Second, "i" enters in the complex wave functions and, in particular, in the spinor wave functions (for fermions), and is related here to electromagnetic interactions. In quantum mechanics, the wave function $\psi(x)$ describing the motion of an electron can be subjected to a position-dependent phase transformation (called a local gauge transformation)

$$\psi(x) \longrightarrow \psi'(x) = e^{-i\alpha(x)}\psi(x), \qquad (1.9)$$

where $\psi(x)$ in the nonrelativistic case is a Schrödinger wave function, in a relativistic case, a spinor wave function. In the dynamic equation of a free electron, in order to compensate for the differentiation of the phase factor $e^{-i\alpha(x)}$ arising from the local gauge transformation, the gradient operator d must be changed to a combination $(d - ieA)$, where e is the charge of an electron and A is called the $U(1)$ gauge potential, i.e., the (external) electromagnetic field. This is the well known gauge principle, or the principle of " minimal coupling ". Here the "i" originates from the fact that the Lie algebra of the $U(1)$ group — the gauge group of electromagnetism — consists of pure imaginary numbers.

The third way for "i" to enter quantum mechanics is through the Clifford algebra $C_{p,q}$. If $\{e_i\}$ is an orthonormal basis of the Euclidean space of dimension $n = p + q$, where p and q are the metric signatures, then the volume element

$$\eta = e_1 e_2 \cdots e_{p+q} \qquad (1.10).$$

has the square

$$\eta^2 = (-1)^{(p-q)(p-q-1)/2}, \qquad (1.11)$$

and is a generator of the dual transformation. For $p-q = 2 \bmod 4$ (or $p-q = 3 \bmod 4$), the square is negative and η belongs to the center of $C^{+}_{p,q}$ or[1] $C_{p,q}$, respectively. For example, in the real Clifford algebra denoted by $\mathcal{R}_{p,q}$, when $p = 3, q = 0$, i.e., the real Pauli algebra[2] $\mathcal{P} = \mathcal{R}_{3,0}$, the volume element $\eta = i$ with

$$i^2 = (\sigma_1\sigma_2\sigma_3)^2 = -1 \qquad (1.12)$$

commutes with all the elements of $\mathcal{P}$. When $p = 1, q = 3$, i.e., the real Dirac algebra $\mathcal{R}_{1,3}$, the volume element $\eta = \gamma_5$ with

$$(\gamma_5)^2 = (\gamma_1\gamma_2\gamma_3\gamma_4)^2 = -1, \qquad (1.13)$$

which commutes with all the elements of even subalgebra $\mathcal{R}^{+}_{1,3}$ of $\mathcal{R}_{1,3}$.

It is not trivial or obvious that the three "i"'s with square root of minus one (quantum-mechanical, electromagnetic and Clifford) in quantum mechanics are one and the same[3]. If they are, the "i", associated with $U(1)$ gauge transformations in the phase factor is the same as the "i" of the volume element, which is associated with the dual transformations in the Clifford algebra; and it must also be consistent with the "i" associated with the differential operators P_x or[3] X. However, the Dirac equation, which is required to be first order in time and relativistically covariant,

---

1 If $p-q = 2 \bmod 4$, n = p+q = even, then the volume element does not belong to the center of the algebra, but belongs to center of its even subalgebra; if $p-q = 3 \bmod 4$, n = p + q = odd, then its volume element belongs to the center of the algebra.

2 In this thesis afterward, I use the term Pauli algebra to mean the real Pauli algebra.

3 The "i" connected to P_x or X depends on the representations.

combines these into one single equation, whose successes indicate that the three "i"'s may be the same in the Clifford algebra. Similar observations have recently been made by Dr. Chen Ning Yang.

Recently, interest has been expressed in the use of Clifford algebras as a unified language for mathematics and physics[5,6]. It integrates the real numbers, complex numbers, quaternions, vectors, tensor, matrix, differential forms, and seems to provide an adequate algebraic structure for all purposes of geometry and physics. Its applications have been developed for mechanics, electromagnetism, special relativity, field theory and mathematical physics[7,8,9,10,11,12,13,14,15]. Especially the Pauli algebra, denoted by $\mathcal{P}$, is a Clifford algebra of 3-dimensional Euclidean space i.e., $\mathcal{R}_{3,0}$. It has been applied to Newtonian mechanics, providing compact methods for vector (including complex vector), matrix and rotation descriptions[8]; to relativistic mechanics, providing a natural framework for a component-free covariant treatment of problems in special relativity [14]with the same power as that of D. Hestenes[6]; to the generalized electromagnetic theory with monopoles, providing a simple, completely symmetric and consistent description[1]; to the Dirac equation, for pursuing a deep understanding of the Dirac equation from both classical and quantum aspects; to the weak and electromagnetic theory[16];and to the general relativity[17].

On the other hand, J.V.Neumann and G.D.Birkhoff[18]proved that one can construct many different models for a propositional calculus in quantum mechanics which can not be differentiated by any known criteria. The Hilbert space $\mathcal{H}$ can be constructed over any field $\mathcal{F}$ having an anti-automorphism satisfying non-negative properties. Up to now, Hilbert spaces $\mathcal{H}$ have been constructed over the following fields:

---

1 " The Pauli algebra approach to generalized electromagnetic theory " by Jiansu Wei and W. E. Baylis (in preparation).

$\mathcal{R}$, real numbers;

C, the complex numbers; and

Q, the quaternions.

In these fields, the Schrödinger equation is generalized to

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \; - - \rightarrow \; \eta\hbar\frac{\partial\psi}{\partial t} = H\psi, \qquad (1.14)$$

where the η takes the form:

$\mathcal{R}$: $\eta = J$;

C: $\eta = i$;

Q: η is one of the three imaginary units (i, j, k).

The relation between complex and real QM has been extensively investigated by Stueckelberg and collaborators[19]and now is understood: complex QM is completely equivalent to the real QM plus a superselection rule: all the observables in real QM described by symmetric operators[1] must commute with a fixed linear operator J, which is antisymmetric (J^t = -J) with J^2 = -1 and anti-commutes with the time-reversal operations. The operator J takes the role of the imaginary number "i" and is nothing other than a real matrix representation of "i". The operator J is necessary in order to have an Uncertainty Principle or the commutation relation. It is also required in order to get

$$\frac{d}{dt}\int d^3x \, |\psi|^2 = 0, \qquad (1.15)$$

i.e., conservation of particles, where ψ has nontrivial dynamics from the Schrödinger eq.(1.14). While the complex and real QM's are simply related, quaternion QM has

1 For here the hermitean conjugate is just a transpose because there are no complex numbers.

many new features which make it a much richer theory[19,20]. Because the quaternion algebra is not commutative, the η is not a linear operator and the Schrödinger eq.(1.14) makes no sense. To overcome this difficulty, the colinear and co-unitary methods are introduced, which allow quaternions to be linear operators[1] acting on rays in Hilbert space. As a result, the gap between quaternion QM and classical physics is greater than the gap between complex QM and classical physics. In classical physics there are no phase relations to be considered when the systems are embedded as subsystems in a larger system, either by adding or multiplying their phase-spaces. In complex QM there are phase-space relations between states that are important when sums are formed, but not when products are formed. In quaternion QM the phase relations are important when states are either added or multiplied.

D. Hestenes[7,12] used the real Dirac algebra $\mathcal{R}_{1,3}$ (called "spacetime geometry"), to elucidate the "difference between a mathematical concept and its representation by symbols" and to uncover a hidden geometric structure of the Dirac equation, especially the geometrical meanings of the "i". Since in the real $\mathcal{R}_{1,3}$, the center $\mathcal{Z}$ is composed only of scalar numbers, the volume element $i = \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ does not belong to $\mathcal{Z}$; it is not compatible with the imaginary number $i = (-1)^{\frac{1}{2}}$ which commutes with all the γ_μ in complex QM. To handle this problem, he defined a right multiplication by "$\underline{i}$" as

$$\underline{i}\psi = \psi i, \tag{1.16}$$

---

1 q as a linear operator, because

$$(\psi a)^{\wedge} q = \psi^{\wedge} a^{\wedge} q = \psi^{\wedge}(qaq^{-1})q = (\psi^{\wedge} q)a,$$

where $q\psi = \psi^{\wedge} q$, $\psi^{\wedge} = q\psi q^{-1}$.

to make "i" a linear operator. When one goes from the complex to the real representation, the unit imaginary "i'" is replaced by the spacelike bivector $i\sigma_3 = \gamma_2\gamma_1$. In this framework, "$i$'" has explicit geometrical and physical meanings:

1). $i' = i\sigma_3$, , with $i'^2 = -1$ is a spacelike bivector;

2). $i' = i\sigma_3$, is the generator of a rotation in the s-plane related to the spin;

3). $i' = i\sigma_3$, is the generator of electromagnetic gauge transformations.

In the Pauli algebra, $\mathcal{P} = \mathcal{R}_{3,0}$ for $p + q = 3 + 0 = 3$, an odd number, and $p - q = 3 - 0 = 3$, so that the center $\mathcal{Z}$ is composed of $\mathcal{P}_0$ and $\mathcal{P}_3$, $\mathcal{Z} = (\mathcal{P}_0 \oplus \mathcal{P}_3)$. Thus, the volume element, $\eta = i = \sigma_1\sigma_2\sigma_3$, $i^2 = -1$, commutes with all the elements of $\mathcal{P}$ as does the imaginary number "i". On the other hand, the complex numbers and the quaternions are subalgebras of Pauli algebra. Therefore $\mathcal{P}$ may be a suitable candidate for unifying the three "i" (quantum-mechanical, electromagnetic and Clifford) in quantum theory.

In this thesis, the Pauli algebra $\mathcal{P}$ is used to construct a real non-relativistic Quantum Mechanics (Schrödinger theory for spinless particles); the fully relativistic description of the Dirac equation for a spinor particle has been given by Baylis[15]. In the Pauli algebra, in order to realize the non-negative properties of the norm of the state vector in Hilbert space, I find that the fields on which the Hilbert space can be constructed are limited to three types. In this thesis, I am interested in the field over the center $\mathcal{Z}$ of $\mathcal{P}$ which is isomorphic to the field of complex numbers. In some sense, this scheme is equivalent to complex QM. Because there are no imaginary numbers in this scheme, the hermitean conjugation has to be replaced by just the transpose which is the operation that changes the order of the operators or the elements of the algebra, and thus all the observables represented by the hermitean operators are symmetric quantities. An "anti-linear" transformation can be realized either by the

transpose, an anti-automorphism which changes the sign of "i", or by an anti-linear transformation, which also changes the sign of "i", but is an automorphism. The time-reversai transformation is constructed as the combination of an anti-automorphism —— a transpose changing the sign of "i" —— and a unitary transformation changing the sign of t. For the parity transformation in QM, there are two transformations from which to choose: the spatial inversion, which is an anti-linear automorphism, and the spatial reversal, which is an anti-automorphism, but —— because it is the product of two anti-linear transformations —— a linear transformation. In this thesis, I find that it is more consistent to choose the spatial reversal as the parity transformation in real frame than the spatial inversion.

In the real quantum mechanics over the field of Z, the "i" has geometric and physical meanings:

1). $i \in Z = (\mathcal{P}_0 \oplus \mathcal{P}_3)$, $i^2 = -1$, commutes with all the other elements, and therefore has the properties usually associated with the imaginary number i;

2). as an anti-hermitean (anti-symmetric) operator, $i^* = -i$, it realizes the Uncertainty Principle and is associated with the differential operator P_x or X;

3). as a pseudoscalar, $\tilde{i} = -i$, it changes sign under spatial inversion.

4). $\bar{i} = (\tilde{i})^* = (\tilde{i^*}) = i$, i.e., under the spatial reversal transformation, it doesn't change sign, which makes it consistent with the parity transformation in classical mechanics as well as in complex QM;

5). $T(i) = i^* = -i$, i.e., under time reversal, it changes sign, and the anti-linearity is attained by an anti-automorphism where that in complex QM is given by an automorphic transformation, namely complex conjugation;

6). i is a generator of the $U(1)$ gauge transformations as well as a generator of dual transformations in the Pauli algebra.

This thesis mainly focuses on those concepts related to the transformation properties of "i" which are different from those in complex QM, such as the time reversal transformation, the parity transformation and the commutation relations.

The thesis is organized as follows: In order to make this thesis self-contained, chapter 2 briefly reviews the real Clifford algebra. It also defines the three maps which are needed later and explains the geometric meanings of the elements of the Pauli algebra. Chapter 3 discusses the construction of Hilbert space and the operators acting on it in real QM for Schrödinger theory. First it is seen that to ensure the non-negative property of inner products in Hilbert space, there are three types of fields over the Pauli algebra on which the Hilbert space can be constructed. The focus here is on the field consisting of the center $\mathcal{Z}$ of $\mathcal{P}$. Then the Hilbert space over the field of complex numbers is translated into the real Hilbert space over the field of the center of the Pauli algebra[1]. For the operators, hermitean conjugation is just a transpose, so that hermitean operators (and thus all observables) are symmetric. In this scheme, because "i" here is not an imaginary number, the anti-linearity can be carried out by either of two methods: an anti-automorphic transpose operation or an automorphic anti-linear operation.

1 Within the real Hilbert space, a state vector is valued on scalars and pseudoscalars (trivectors) in the Pauli algebra.

In chapter 4 the real non-relativistic QM for spinless particles (Schrödinger theory) is constructed. This chapter focuses on the realizations and interpretations of the commutation relation, the time-reversal transformation and spatial transformations. For the commutation relation, the pseudoscalar "i" realizes the Uncertainty Principle with all the properties of the imaginary number "i". For the time-reversal transformation, " + " instead of " * " executes the anti-linearity for the time change, but here " + " is an anti-automorphism so that its transformation must be performed carefully within the inner products or when acting on the states in some specified representation.

For the parity transformation, the situation is somewhat complicated. In the first important place when one tries to perform spatial-inversion transformations, it is sometimes to change from a passive transformation (changing the basis vectors) to an active transformation (fixing the basis, changing the components), and one must keep the sign change of "i", which leads to spatial-inversion transformations with the property of the anti-linear transformation. Because it is an automorphism, when applied to eq.(1.7), it implies that P_x and X have different transformation properties, and therefore that spatial inversion as a parity transformation is not consistent with complex QM or classical physics. However, the spatial-reversal transformation can be constructed as a parity transformation, where the spatial-reversal transformation is an anti-automorphism, but also a linear transformation composed of two anti-linear transformations. Since spatial reversal is an anti-automorphism, its application must be made with care. Finally the basic transformation rules for some basic physical variables are examined.

In chapter 5, the parity transformation is applied to get the superselection rules, the spatial-reversal operator is interpreted as a constant of motion, and the real scalar

representation is constructed. Furthermore, $U(1)$ gauge transformations in which the phase factor is pseudoscalar-valued are found. Chapter 6 contains a summary and discussion.

CHAPTER 2. THE REAL PAULI ALGEBRA

2.1 The Real Clifford Algebra

The real Clifford algebra can be constructed as a quotient of the tensor algebra over the field of real numbers. If we assume that the vector space $\mathcal{V}$ has a real-valued metric g, and let I be the ideal of the tensor algebra $T(\mathcal{V})$, consisting of sums of terms of the form[10]

$$a \otimes \{x \otimes x - g(x,x)\} \otimes b,$$
$$a, b \in T(\mathcal{V}), x \in \mathcal{V}, \tag{2.1}$$

then the real Clifford algebra associated with $\mathcal{V}$ is[1]

$$\mathcal{R}(\mathcal{V}, g) = T(\mathcal{V})/I. \tag{2.2}$$

From this definition, it is easy to see that because the ideal I is not a Z-graded subspace, then neither is $\mathcal{R}(\mathcal{V}, g)$, although it can be decomposed into Z-homogeneous subspaces:

$$\mathcal{R}(\mathcal{V}, g) = \sum_{p=1}^{n} \mathcal{P}_p(\mathcal{R}(\mathcal{V}, g)) = \sum_{p=1}^{n} \mathcal{R}_p \tag{2.3}$$

where n is the dimension of $\mathcal{V}$ and the projection operator $\mathcal{P}_p$ projects $\mathcal{R}(\mathcal{V}, g)$ onto the homogeneous subspace $\mathcal{R}_p$ of p-forms.

Generally, for a p-form ω in the subspace $\mathcal{R}_p$ and $x \in \mathcal{V}$, the Clifford product is given by

$$x\omega = x \wedge \omega + i_x \omega, \tag{2.4}$$

---

1 If the metric term in eq.(2.1) omitted, then this is a definition of the exterior algebra.

where $i_{\tilde{x}}$ is an interior derivative with respect to $\tilde{x} \in \mathcal{V}^*$, the dual space of $\mathcal{V}$. For example, if $x, y \in \mathcal{V}$, then the Clifford product has an explicit form in terms of exterior and interior products[1]

$$xy = x \wedge y + g(x, y). \tag{2.5}$$

For any vector space $\mathcal{V}$ with a basis $\{e_\mu, \mu = 1, 2, \cdots, n\}$, where e_μ obey the relations:

$$(e_\mu)^2 = 1, \mu \in \{1, 2, \cdots, p\}$$

$$(e_\mu)^2 = -1, \mu \in \{p+1, p+2, \cdots, p+q\}$$

$$e_\mu e_\nu + e_\nu e_\mu = 0, \mu \neq \nu, \tag{2.6}$$

we can use the formula (2.4) to construct the multiplication table for any real Clifford algebra

$$\mathcal{R}(\mathcal{V}, g) = \mathcal{R}_{p,q} \tag{2.7}$$

with the basis

$$\{1, e_\mu, e_\mu e_\nu, e_\mu e_\nu e_\rho, \cdots, e_1 e_2 \cdots e_n\} \tag{2.8}$$

$$1 \leq \mu < \nu < \rho < \cdots \leq n.$$

---

1 For

$$(x+y) \otimes (x+y) = x \otimes x + x \otimes y + y \otimes x + y \otimes y$$

$$g(x+y, x+y) = g(x, x) + 2g(x, y) + g(y, y)$$

and therefore

$$x \otimes y = x \wedge y + g(x, y)$$

$$+ \frac{1}{2}\{[(x+y) \otimes (x+y) - g(x+y, x+y)] - [x \otimes x - g(x, x)] - [y \otimes y - g(y, y)]\}$$

where the term in {} is an element of I, $\otimes$ is the tensor product, and $\wedge$ is the exterior product:

$$x \wedge y = 1/2(x \otimes y - y \otimes x)$$

The dimension of $\mathcal{R}_{p,q}$ is therefore $\sum_{m=0}^{n}\binom{n}{m} = (1+1)^n = 2^n$.

In terms of the structures of $\mathcal{R}_{1,0}, \mathcal{R}_{1,1}, \mathcal{R}_{0,q}$ for $q = 1, 2, 3, 4$, the following relations determine the structure of all the real Clifford algebras[1][10]:

$$\mathcal{R}_{p+1,q} \cong \mathcal{R}_{q+1,p}$$

$$\mathcal{R}_{p+1,q+1} \cong \mathcal{R}_{q,p} \otimes \mathcal{R}_{1,1}$$

$$\mathcal{R}_{p,q+4} \cong \mathcal{R}_{q,p} \otimes \mathcal{R}_{0,4}. \qquad (2.9)$$

The Z_2gradation[2] of the Clifford algebra $\mathcal{R}_{p,q}$ ensures that the elements of even degree form a subalgebra $\mathcal{R}^{+}_{p,q}$ which has the relation:

$$\mathcal{R}^{+}_{p,q} \cong \mathcal{R}_{q,p-1} \qquad (2.10)$$

The center of $\mathcal{R}_{p,q}$ is the subalgebra $\mathcal{Z}$ consisting of those elements of $\mathcal{R}_{p,q}$ which commute with all elements of $\mathcal{R}_{p,q}$. For $\mathcal{R}_{p,q}$, if p + q = n even, then $\mathcal{Z} = \mathcal{R}_0$; if p + q = n odd, then $\mathcal{Z} = (\mathcal{R}_0 \oplus \mathcal{R}_n)$.

For the Clifford algebra $\mathcal{R}_{p,q}$ there are three automorphisms and anti-automorphisms[3]:

1). The main automorphism is defined by a map " ~ ", which is connected to the spatial reflection of vector space $\mathcal{V}$:

$$e_i \mid - - \rightarrow \tilde{e}_i = -e_i, \qquad (2.11)$$

---

1 $\mathcal{R}_{1,0} \cong \mathcal{R} \oplus \mathcal{R}, \mathcal{R}_{1,1} \cong \mathcal{M}_2(R), \mathcal{R}_{0,1} \cong C(R), \mathcal{R}_{0,2} \cong \mathcal{H}(R),$

$$\mathcal{R}_{0,3} \cong \mathcal{H}(R) \oplus \mathcal{H}(R), \mathcal{R}_{0,4} \cong \mathcal{H}(R) \otimes \mathcal{M}_2(R),$$

2 It is easy to see: $\mathcal{R}_{even}\mathcal{R}_{even} = \mathcal{R}_{even}$, $\mathcal{R}_{odd}\mathcal{R}_{even} = \mathcal{R}_{odd}$, $\mathcal{R}_{odd}\mathcal{R}_{odd} = \mathcal{R}_{even}$.

3 There are only two independent maps.

such that

$$\tilde{a} = (-1)^p a, \qquad \forall a \in \mathcal{R}_p. \tag{2.12.1}$$

$$(ab)\tilde{} = \tilde{a}\tilde{b}, \qquad \forall a, b \in \mathcal{R}_{p,q} \tag{2.12.2}$$

$$\tilde{\tilde{a}} = a, \qquad \forall a \in R_{p,q}, \tag{2.12.3}$$

2). The reverse map (later called a hermitean transpose), which is an anti-automorphism, is defined by map " + ":

If $e_a = e_{i_1} e_{i_2} \cdots e_{i_p}$, then

$$e_a^* = e_{i_p} e_{i_{p-1}} \cdots e_{i_1} = (-1)^{\frac{1}{2}p(p-1)} e_a \tag{2.13}$$

and $a \in \mathcal{R}_p$

$$a \mid - - \rightarrow a^* = (-1)^{\frac{1}{2}p(p-1)} a, \tag{2.14}$$

i.e.,

$$a^* = \begin{cases} a, & if \quad p = 0 \mod 4 \quad or \quad 1 \mod 4 \\ -a, & if \quad p = 2 \mod 4 \quad or \quad 3 \mod 4 \end{cases}$$

It is obvious that

$$(ab)^* = b^* a^*, \qquad a, b \in \mathcal{R}_{p,q}, \tag{2.15.1}$$

$$(a^*)^* = a. \tag{2.15.2}$$

3). The product of 1) and 2) in either order is an anti-automorphic map " - ":

$$\bar{a} = (\tilde{a})^* = (\tilde{a^*}) = (-1)^{\frac{1}{2}p(p+1)} a. \qquad a \in \mathcal{R}_p \tag{2.16}$$

It is obvious that

$$\overline{(ab)} = \bar{b}\bar{a}, \qquad a, b \in \mathcal{R}_{p,q}, \tag{2.17.1}$$

$$\overline{(\bar{a})} = a. \tag{2.17.2}$$

2.2 Real Pauli Algebra

If the vector space $\mathcal{V}$ is 3-dimensional with the basis[1] $\{\sigma_1, \sigma_2, \sigma_3\}$, associated with the Euclidean metric p = 3, q = 0, then the real Clifford algebra $\mathcal{R}_{3,0}$, called the Pauli algebra, denoted by $\mathcal{P}$, has a basis

$$\{1, \sigma_i, \sigma_i\sigma_j, i = \sigma_1\sigma_2\sigma_3\} \qquad (2.18)$$

$$1 \le i < j \le 3$$

The center $\mathcal{Z}$ of $\mathcal{P}$ is composed of the set ($\mathcal{P}_0 \oplus \mathcal{P}_3$), where i belongs to $\mathcal{Z}$, so i commutes with all the elements of $\mathcal{P}$. The dual exchange can be simply realized by the geometric products with the algebra element i, so the basis can be rewritten as

$$\{1, \sigma_i, i\sigma_i, i\} \qquad (2.19)$$

consisting of

one scalar (1),

three basis vectors $(\sigma_1, \sigma_2, \sigma_3)$,

three bivectors $(i\sigma_1, i\sigma_2, i\sigma_3)$, and

one trivector (i), also called a pseudoscalar,

where $i\sigma_i$ are the basis vectors dual to σ_i, while i is the dual of 1.

The sets of $\{1\}, \{\sigma_i\}, \{i\sigma_i\}, \{i\}$ span the real linear subspaces $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ of $\mathcal{P}$, respectively. The Pauli algebra is a direct sum:

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 \qquad (2.20)$$

of the homogeneous k-vector spaces.

The general element of $\mathcal{P}$ can be displayed as :

---

1 Where σ is used as usual as the basis of Pauli algebra.

$$a = a_0 + \frac{1}{1!}a_1^i\sigma_i + \frac{1}{2!}a_2^{ij}\sigma_i \wedge \sigma_j + \frac{1}{3!}a_3^{ijk}\sigma_i \wedge \sigma_j \wedge \sigma_k$$

$$= a_0 + a_1^i\sigma_i + a_2^i i\sigma_i + a_3 i = a_0 + \vec{a}_1 + i\vec{a}_2 + ia_3 \qquad (2.21)$$

where $a_2^i = \frac{1}{2}\epsilon_{jk}a_2^{jk}$, $a_3 = \frac{1}{3!}\epsilon_{ijk}a^{ijk}$, dim $a_0 = 1$, dim $a_1 = 3$, dim $a_2 = 3$, dim $a_3 = 1$.

In the Pauli algebra $\mathcal{P}$, the three maps are explicitly realized:

1. The main automorphism " $\sim$ ", called spatial inversion:

$$\tilde{a} = a_0 - \vec{a}_1 + i\vec{a}_2 - ia_3 \qquad (2.22)$$

which changes the signs of i and the vectors, and which really describes a spatial reflection for a real linear space. This is different from the spatial reflection in complex vector space if we consider the "i" as an imaginary number instead of as a pseudoscalar.

2. The reversal map (or transpose) " + ", called hermitean conjugation:

$$a^* = a_0 + \vec{a}_1 - i\vec{a}_2 - ia_3 \qquad (2.23)$$

which is isomorphic to the complex conjugation if we try to interpret a as a complex general vector, " + " only changes the sign of i.

3.The product map " - " of " + " and " $\sim$ ", called spatial reversion:

$$\bar{a} = a_0 - \vec{a}_1 - i\vec{a}_2 + ia_3 \qquad (2.24)$$

the i does not change its sign under the map " - ", while a vector does. This makes sense if one interprets a as a general complex vector. Then the spatial reversal as a parity transformation only changes the sign of vectors, but not the sign of i.

In chapter 4, the reader will see how these maps are connected with the time-reversal and parity transformations in real QM.

The Pauli algebra, as the even subalgebra of the Dirac algebra (called the space-time algebra), which describes the 4-dimensional space-time of relativity, is called the space algebra and describes 3-dimensional Euclidean space. Baylis and Jones[14]defined a general element

$$a = a_0 + \vec{a} \tag{2.25}$$

and the product[1]

$$a\bar{a} = \bar{a}a = a^2 - \vec{a}^2 \tag{2.26}$$

which suggests the structure of Minkowskian 4-dimensional space time. Applications can be found in Ref.[14].

2.3 Geometric Interpretation

2.3.1 Vectors and Their Products.

Let $\vec{a}, \vec{b} \in \mathcal{P}_1$, then

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b} \tag{2.27}$$

gives full relations between two vectors, where the scalar part

$$\vec{a} \cdot \vec{b} = (\vec{a}\vec{b} + \vec{b}\vec{a})/2 \tag{2.28}$$

gives the projection of $\vec{a}$ on $\vec{b}$ or of $\vec{b}$ on $\vec{a}$; while the antisymmetric part

---

1 There are a couple of reasons for choosing the product $a\bar{a}$ and not $a\tilde{a}$ as a realization of the Minkowskian metric: for most physical cases, there is no difference, but if one generalizes a to an element including scalar, pseudoscalar, vector and bivector parts, then $a\bar{a}$ is a Lorentz scalar but $a\tilde{a}$ not; if one applies it to the generalized electromagnetic theory with monopoles, the $a\bar{a}$ definition gives a consistent description for this theory but $a\tilde{a}$ does not.

$$\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a} = (\vec{a}\vec{b} - \vec{b}\vec{a})/2 = |\vec{a}||\vec{b}|\sin(\theta)(i\hat{\theta}) \qquad (2.29)$$

gives the oriented plane formed by $\vec{a}$ and $\vec{b}$, $i\hat{\theta}$ describes the orientation of the plane, with $\hat{\theta}$ being the normal direction to the plane.

Let $\vec{a}, \vec{b}, \vec{c} \in \mathcal{P}_1$, then

$$\vec{a}\vec{b}\vec{c} = \vec{a}(\vec{b}\cdot\vec{c} + \vec{b}\wedge\vec{c}) = (\vec{b}\cdot\vec{c})\vec{a} + \vec{a}\cdot(\vec{b}\wedge\vec{c}) + \vec{a}\wedge\vec{b}\wedge\vec{c} \qquad (2.30)$$

where

$$\vec{a}\wedge\vec{b}\wedge\vec{c} = \vec{a}\wedge(\vec{b}\wedge\vec{c}) = (\vec{a}\wedge\vec{b})\wedge\vec{c} = i\{\vec{a}\cdot(\vec{b}\times\vec{c})\} = i\{(\vec{a}\times\vec{b})\cdot\vec{c}\}$$

is a pseudoscalar, representing a directed volume.

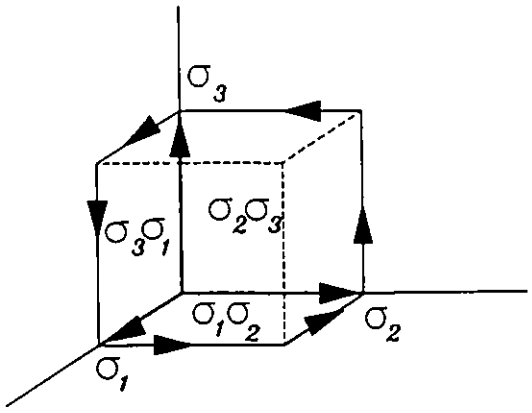

Fig.1. An orthonormal basis for $\mathcal{P}$ in 3-dimensional Euclidean space. The directed lines represent unit vectors $\{\sigma_i\}$. The directed planes represent unit bivectors $\{\sigma_i\sigma_j; i \neq j\}$. The oriented cube represents the unit pseudoscalar $i = \sigma_1\sigma_2\sigma_3$.

2.3.2 Transformations.

A general element of transformation in the Pauli algebra can be written as:

$$T = b_0 e^{\vec{\omega}/2 + i\vec{\theta}/2 + i\phi/2} \tag{2.31}$$

where each term has the geometrical meaning:

b_0: a dilation;

$\vec{\omega}$: a pure boost;

$i\vec{\theta}$: a pure rotation;

$i\phi$: a dual transformation or a $U(1)$ gauge transformation.

In $\mathcal{P}$ space, b_0 acting on any element just increases its components by the factor b_0. The restricted Lorentz transformation, a combination of a pure boost and a pure rotation, on a 4-vector a is

$$L: \quad a \dashrightarrow a' = LaL^{\dagger} \tag{2.32}$$

where $L = e^{\vec{\omega}/2 + i\theta/2}$. The realization of dual transformations

$$R = e^{-i\frac{\phi}{2}} = \cos\frac{1}{2}\phi - i\sin\frac{1}{2}\phi \tag{4.33}$$

is simply applied to any element a by

$$a^R = RaR = e^{-i\phi}a. \tag{4.34}$$

Here $e^{-i\phi}$ also defines a $U(1)$ Lie group, acting on both the operator space and the Hilbert space.

2.4 The Spinor Space of $\mathcal{P}$

The spinor space of three dimensional Euclidean space is a minimal left ideal $\mathcal{S}$ of the Clifford algebra $\mathcal{P}$, where $\mathcal{S}$ consists of $\mathcal{S} = \mathcal{P}e$ and e is a primitive idempotent[1]. For $\mathcal{P}$, p=3, q=0, $k = 0 - r_{-3} = -(r_5 - 4) = -(3-4) = 1$, so $e = \frac{1}{2}(1 + \sigma_3)$ is a primitive idempotent. The spinor space $\mathcal{S}$ has a basis

$$f_1 = e = \frac{1}{2}(1 + \sigma_3) \tag{2.35.1}$$

$$f_2 = \sigma_1 e = \frac{1}{2}(\sigma_1 - \sigma_3\sigma_1) \tag{2.35.2}$$

In terms of the matrix representations of $\sigma_1, \sigma_2, \sigma_3$ as the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.36}$$

f_1, f_2 are represented by

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.37}$$

and any spinor can be expanded on this basis

$$\psi = \psi_1 f_1 + \psi_2 f_2 = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}, \tag{2.38}$$

where $\psi_1, \psi_2 \in Z$, the center of $\mathcal{P}$.

---

1 The number of independent primitive idempotents is given by 2^k where $k = q - r_{q-p}$ and r_{q-p} is the Radon-Hurwitz number r_i for $i \in Z$, related by the recursion formula $r_{i+8} = r_i + 4$. Any e can be written in the form:

$$e = \frac{1}{2}(1 \pm e_{i_1})\frac{1}{2}(1 \pm e_{i_2})\cdots\frac{1}{2}(1 \pm e_{i_k}).$$

CHAPTER 3. THE REAL HILBERT SPACE

In this chapter, two different linear spaces are distinguished, the linear space of the real Clifford algebra whose subsets are used to construct the field $\mathcal{F}$, and the vector space $\mathcal{V}_n$ which is used to construct the real Hilbert space.

In order to construct the Hilbert space, the first step is to define the field $\mathcal{F}$ over the Pauli algebra, and then to construct the Hilbert space over the field of $\mathcal{F}$.

3.1 A Field $\mathcal{F}$ Over the Pauli Algebra

A field $\mathcal{F}$ is a commutative ring which contains for each element $a \neq 0$ an " inverse " element a^{-1} satisfying the equation $a^{-1}a = 1$. If the ring is not commutative, then $\mathcal{F}$ is called a skew field[21].

In order to realize the non-negative property of inner products on the Hilbert space, it is easy to prove that there are only three possible types of field which can be constructed over the real Pauli algebra, i.e.:

1. $\mathcal{F}_1 = \{a \mid a \in \mathcal{P}_0\}$;
2. $\mathcal{F}_2 = \{a \mid a \in (\mathcal{P}_0 \oplus \mathcal{P}_2)\}$;
3. $\mathcal{F}_3 = \{a \mid a \in (\mathcal{P}_0 \oplus \mathcal{P}_3)\}$.

For a simple proof, let the " inner product " on the subset $\mathcal{F}$ of $\mathcal{P}$, namely

$$(a, b) = a^{*}b, \qquad a, b \in \mathcal{F} \tag{3.1}$$

satisfy, $\forall a, b, c \in \mathcal{F}$

$$(a, a) \geq 0 \tag{3.2.1}$$

$$(a, b) = (b, a)^{*} \tag{3.2.2}$$

$$(a, bc) = (a, b)c \tag{3.2.3}$$

$$(a, b + c) = (a, b) + (a, c). \tag{3.2.4}$$

In order to satisfy the first non-negative property, with the three maps described in chapter 2, one notes[1]:

$\forall a \in \mathcal{P}$,

$$(a \pm \tilde{a}, a \pm \tilde{a}) = (a \pm \tilde{a})^{*}(a \pm \tilde{a}) \geq 0. \tag{3.3}$$

Thus $a \pm \tilde{a}$ satisfy (3.1). Factoring the i out of $\{a - \tilde{a}, a \in \mathcal{P}\}$, $\mathcal{F}$ could be chosen as:

1. $\mathcal{F}_1 = \mathcal{P}_0$;

2. $\mathcal{F}_2 = \mathcal{P}_0 \oplus \mathcal{P}_2$;

Further, $\forall a \in \mathcal{P}$,

$$(a + \bar{a}, a + \bar{a}) = (a + \bar{a})^{*}(a + \bar{a}) \geq 0 \tag{3.4}$$

and thus $\mathcal{F} = \{a + \bar{a}, a \in \mathcal{P}\}$ also satisfies (3.1). Therefore

3. $\mathcal{F}_3 = \mathcal{P}_0 \oplus \mathcal{P}_3$

Here $\mathcal{F}_1$ is nothing but the field of real numbers, whereas $\mathcal{F}_2$ is a skew field, isomorphic to the field of quaternions and $\mathcal{F}_3$ is isomorphic to the field of complex numbers[2], In this thesis, the focus is on the third type of field, defined by:

$$\mathcal{F}_{\mathcal{P}_0 \oplus \mathcal{P}_3} = \{a \mid a \in (\mathcal{P}_0 \oplus \mathcal{P}_3)\} \tag{3.5}$$

---

1 It is easy to verify that the following three subsets satisfy the definition of field.

2 Here "isomorphic" just refers to an algebraic isomorphism, i.e., the same algebraic structures, but the "i" in these two schemes have different geometric meanings.

Although every element in $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ can be written as $\alpha = \alpha_0 + i\alpha_3$, where α_0 and α_3 are real numbers, and it is symbolically the same as the complex numbers, the "i" has a different meaning from the imaginary number "i". The "i" in the Pauli algebra is a volume element of $\mathcal{P}$ in the center $\mathcal{Z} = (\mathcal{P}_0 \oplus \mathcal{P}_3)$, which commutes with all the elements of $\mathcal{P}$, and it is a pseudoscalar, or a unit trivector with $i^2 = -1$. Under the inverse map, as in the complex conjugation in complex numbers, its sign changes. This is important in quantum mechanics for the realization of the hermitean conjugate transformations, the Uncertainty Principle and the time-reversal transformation. The "i" in the Pauli algebra has geometrical meaning, since on the one hand, under the spatial inversion it changes its sign, whereas the "i" in complex numbers does not. On the other hand, under spatial reversal, the "i" does not change its sign, whereas the vectors and bivectors do and this results in very different realizations and interpretations for parity transformations in the real QM over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ from that in complex QM. The "i" in real quantum theory on $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ is not introduced externally, but arises automatically as an outer product of the three basis vectors of Euclidean space.

3.2 A Real Vector Space Over the Field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$

The following subsection defines a vector space over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$. It is a trivial procedure which just translates the vector space over the field of complex numbers into the vector space over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$. The development follows Ref.[22,23].

A vector space $\mathcal{V}_{\mathcal{H}}$ over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ with operations of addition and multiplication by scalars[1] has the following definitions:

The operation of vector addition is a map

$$\begin{aligned} &(f, g) - - \rightarrow f + g, \\ &(f, g) \in \mathcal{V}_{\mathcal{H}} \times \mathcal{V}_{\mathcal{H}}, \\ &f + g \in \mathcal{V}_{\mathcal{H}}, \end{aligned} \tag{3.6}$$

of $\mathcal{V}_{\mathcal{H}} \times \mathcal{V}_{\mathcal{H}}$ into $\mathcal{V}_{\mathcal{H}}$, while the operation of multiplication by a scalar α from the field $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ is a map

$$\begin{aligned} &(\alpha, f) - - \rightarrow \alpha f, \\ &(\alpha, f) \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3} \times \mathcal{V}_{\mathcal{H}}, \\ &\alpha f \in \mathcal{V}_{\mathcal{H}}, \end{aligned} \tag{3.7}$$

where the important thing is that, since $\alpha \in Z, \alpha f = f \alpha$.

These two operations are required to satisfy the following axioms for $f, g, h \in \mathcal{V}_{\mathcal{H}}$ and $\alpha, b \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$:

(1). $f + g = g + f$, commutativity of vector addition;

(2). $(f + g) + h = f + (g + h)$, associativity of vector addition;

(3). $\exists 0 \in \mathcal{V}_{\mathcal{H}}$, $\ni g$ satisfies $(f + g) = f$, iff $g = 0$;

(4). $\alpha(f + g) = \alpha f + \alpha g$, distributivity over $\mathcal{V}_{\mathcal{H}}$;

(5). $(\alpha + b) f = \alpha f + b f$, distributivity over $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$;

(6). $(\alpha b) f = \alpha (b f)$, associativity over $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$;

(7). There is a unique unit element $1 \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ such that $1 f = f$.

---

1 Where "scalars" do not correspond to scalars in $\mathcal{P}$, which comprise both scalars and trivectors.

Similarly, one can introduce definitions of the linear independence, the dimension and the isomorphism of vector spaces over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ in analogy to those for the vector spaces over the field of real numbers or of complex numbers[22, 23].

3.3 Euclidean (Pre-Hilbert) Space Over the Field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$

A Euclidean space (or pre-Hilbert space) $\mathcal{E}$ is a vector space $\mathcal{V}_{\mathcal{H}}$ on which an inner product is defined[22,23].

An inner product $< \cdot \mid \cdot >$ on the vector space $\mathcal{V}_{\mathcal{H}}$ over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ is defined by a map of the set $\mathcal{V}_{\mathcal{H}} \times \mathcal{V}_{\mathcal{H}}$ into the set $(\mathcal{P}_0 \oplus \mathcal{P}_3)$ of scalars and pseudoscalars:

$$\begin{aligned} &(f, g) \mid - - \rightarrow < f \mid g >, \\ &(f, g) \in \mathcal{V}_{\mathcal{H}} \times \mathcal{V}_{\mathcal{H}}, \\ &< f \mid g > \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}, \end{aligned} \tag{3.8}$$

which satisfies the following requirements:

(1). $< f \mid f >> 0, \forall f \neq 0$, a non-negative form[1].

(2). $< f \mid g > = < g \mid f >^{+}$, hermitean symmetry.

(3). $< f \mid \alpha g > = \alpha < f \mid g >, \alpha \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$

(4). $< f \mid g + h > = < f \mid g > + < f \mid h >$

The second rule is different from that for the inner product defined on the vector space over the field of complex numbers because here the field $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ is real and valued on $\mathcal{P}_0 \oplus \mathcal{P}_3$ since the transpose " + " has taken the role of " * " in complex QM, and it changes the sign of i. In the field of complex numbers, instead of the reversion map " + " used here, complex conjugation " * " is used.

---

1 Here $< f \mid f >$ is a real scalar in $\mathcal{P}$.

From the relations (2), (3) and (4), it is easy to prove:

$$< \alpha f \mid g >= \alpha^{*} < f \mid g > \tag{3.9}$$

$$< f + g \mid h >=< f \mid h > + < g \mid h > \tag{3.10}$$

3.4 Hilbert Space Over the Field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$

In the Pauli algebra, $\forall \alpha \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$,

$$| \alpha |^2 = \alpha^{*} \alpha = \alpha \alpha^{*} \geq 0 \tag{3.11}$$

is a real scalar. For $< f \mid g > \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$,

$$\begin{aligned} |< f \mid g >|^2 &= < f \mid g >^{*} < f \mid g >=< f \mid g >< f \mid g >^{*} \\ &=< g \mid f >< f \mid g >=< f \mid g >< g \mid f > \geq 0 \end{aligned} \tag{3.12}$$

From the definition of the inner product in the previous subsection, for any elements $f, g \in \mathcal{E}$, it is easy to prove the Cauchy-Schwarz inequality:

$$|< f \mid g >|^2 \leq < f \mid f >< g \mid g > \tag{3.13}$$

which is the most important basis of the theory of Hilbert space. To establish (3.13), from property (1) of section 3.3, note that for any $\alpha \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$,

$$< f + \alpha g \mid f + \alpha g > \geq 0 \tag{3.14}$$

One can choose α to be

$$\alpha = -\frac{< g \mid f >}{< g \mid g >}, \tag{3.15}$$

so that substituting into (3.14) gives

$$< g \mid g >< f \mid f > - < f \mid g >< g \mid f > \geq 0. \tag{3.16}$$

This leads to the Cauchy-Schwarz inequality. With this Cauchy-Schwarz inequality, it is easy to define a real scalar valued function

$$|f| = (<f|f>)^{1/2} \tag{3.17}$$

as a norm, which satisfies the following conditions:

1. $|f| > 0, \forall f \neq 0$
2. $|0| = 0$
3. $|\alpha f| = |\alpha|\,|f|$
4. $|f+g| \leq |f| + |g|$

where $f, g \in \mathcal{E}$. The first three properties can be deduced directly from the definition of the inner product. The fourth can be achieved from these two equations:

$$|f+g|^2 = <f+g|f+g> = <f|f> + <g|g> + <f|g> + <g|f>$$
$$= <f|f> + <g|g> + <f|g> + <f|g>^* \tag{3.18}$$

and[1]

$$\frac{1}{2}(<f|g> + <f|g>^*) = |(<f|g>)_0| \leq |<f|g>| \leq |f|\,|g| \tag{3.19}$$

With the norm of the difference of two vectors f and g, i.e., the metric $d(f,g) = |f-g|$, concepts such as convergence and completeness can be defined so that the Euclidean space, which is complete in the norm, is seen to be a Hilbert space $\mathcal{H}$. The other properties of Hilbert space over the field of complex numbers can be easily translated into our real Hilbert space over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$.

3.5 Linear and Antilinear Operators

In quantum mechanics, a linear operator A is defined as follows: if a vector $|\psi> \in \mathcal{H}$, then[2] $A|\psi> \in \mathcal{H}$ and satisfies

---

1 Where $()_0$ means to take the scalar part.

2 In general, $|\psi>$ and $A|\psi>$ may belong to different Hilbert spaces.

$$A(a\,|\,\psi>+b\,|\,\phi>)=a\,A\,|\,\psi>+b\,A\,|\,\phi> \qquad (3.20)$$

for $a, b \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$. From this definition, one sees that the actions of unit basis vectors σ_i on the Hilbert space have no definitions. One only can take the components of the vectors or bivectors as the linear operators acting on Hilbert space.

For an antilinear operator A in real QM over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$, because there is no complex conjugation here, its realization can be achieved by either an automorphism or an anti-automorphism, which can be defined by

$$\begin{aligned} A(a\,|\,\psi>+b\,|\,\phi>) &= A a A^{-1} A(|\,\psi>)+A b A^{-1} A(|\,\phi>) \\ &= a^{*} A(|\,\psi>)+b^{*} A(|\,\phi>) \end{aligned} \qquad (3.21.1)$$

or

$$\begin{aligned} A(a\,|\,\psi>+b\,|\,\phi>) &= A(|\,\psi>)A(a)+A(|\,\phi>)A(b) \\ &= a^{*} A(|\,\psi>)+b^{*} A(|\,\phi>) \end{aligned} \qquad (3.21.2)$$

Here one must note the difference between the Hilbert spaces over fields of complex numbers and those over $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$. For the former, the operation A is an automorphism, while for the latter, the operation A is either an automorphism or an anti-automorphism. In the next chapter, it will be seen that the spatial-inversion transformation is an antilinear operator arising from an antilinear automorphism, while the hermitean conjugate " + " is an antilinear operator associated with a linear anti-automorphism.

For any[1] operators A and B, from the inner product, one has the identities

$$<\psi\,|\,A\phi>=<A^{*}\psi\,|\,\phi> \qquad (3.22.1)$$

$$<A\psi\,|\,\phi>=<\psi\,|\,A^{*}\phi> \qquad (3.22.2)$$

1 Here "any" means any meaningful operators in Hilbert space.

with properties, $\forall \lambda \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$

$$(A^*)^* = A \tag{3.22.3}$$

$$(\lambda A)^* = \lambda^* A^* \tag{3.22.4}$$

$$(A + B)^* = A^* + B^* \tag{3.22.5}$$

A necessary and sufficient condition for an operator A to be self-adjoint[1], called hermitean, is that, for $\forall \mid \psi >, \mid \phi > \in \mathcal{H}$:

$$< \psi \mid A\phi > = < A\psi \mid \phi > \tag{3.23.1}$$

One writes

$$A^* = A. \tag{3.23.2}$$

From this, one sees that all the hermitean operators are symmetric operators, and the expectation value of any hermitean operator is a real scalar number

$$<\mid A \mid>^* = <\mid A^* \mid> = <\mid A \mid> . \tag{3.24}$$

The constant operator i, on the other hand, is anti-hermitean, anti-symmetric.

A necessary and sufficient condition for an operator U to be unitary is that, for $\forall \mid \psi >, \mid \phi > \in \mathcal{H}$:

$$< U\psi \mid U\phi > = < \psi \mid \phi > \tag{3.25.1}$$

$$U^* U = U U^* = 1 \tag{3.25.2}$$

whereas to be anti-unitary

$$< U\psi \mid U\phi > = < \psi \mid \phi >^* = < \phi \mid \psi > . \tag{3.25.3}$$

Here one must notice that the concepts about hermitean and unitary operators have meanings only in relation to an inner product in Hilbert space, i.e., the operator and

---

1 For "adjoint", mathematically, one should note the distinct domains of A and A^*.

its hermitean adjoint act on dual spaces respectively. For example, in the x-representation $P_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$, $P_x = P_x^*$ does not mean that $-\frac{\hbar}{i}\frac{\partial}{\partial x}$ equals to $\frac{\hbar}{i}\frac{\partial}{\partial x}$, it only means that P_x acting on a state vector is equal to P_x^* acting on the dual state vector. This is very important for our scheme for understanding anti-automorphisms such as the time-reversal or spatial-reversal transformations.

All the other concepts corresponding to the operators in complex QM can be easily translated into our scheme.

CHAPTER 4. NON-RELATIVISTIC REAL QUANTUM MECHANICS

This chapter concentrates on non-relativistic real quantum mechanics for spinless particles, especially topics related to "i" with interpretations different from those of complex quantum mechanics. Interpretations which are the same in real and complex quantum mechanics, may be mentioned only briefly, if at all.

4.1 Quantum States, Observables and Operators

In Schrödinger's formulation of quantum mechanics, the state of a point particle is described by a wave function $\psi(x, y, z, t)$ of a position x, y, z and the time t, where the function assumes values from an infinite-dimensional Hilbert space $\mathcal{H}$ (or subspace) over the field[1] $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$. If there are two wave functions ψ_1, ψ_2, such that

$$\psi_1 = c\psi_2, \tag{4.1}$$

where $c \in \mathcal{F}_{\mathcal{P}_0 + \mathcal{P}_3}$, then it is said that they describe the same physical state.

Any quantity that can be measured is called an observable. With this definition, an observable is always connected with a process of a measurement. In QM, the process of any measurement can change a prepared state to another state, corresponding to a linear transformation acting on the associated state vector in the Hilbert space. The initial wave function describes the prepared state of the system, and the transformed wave function specifies the state of the system after the measurement. In all cases, measurements of the physical observables give real quantities (no pseudoscalars). Thus, it follows, consistent with eq.(3.24), that the corresponding operators must be

---

1 In complex Hilbert space, the ψ is a complex-valued function, whereas here, ψ is a real multi-vector(scalar + trivector)-valued function.

hermitean[1]. The complete postulates of quantum mechanics (the same postulates are valid in our real frame as in complex QM, the only differences being those associated with complex conjugation and the spatial transformation) can be found in most quantum mechanics texts[25,26,27].

4.2 The Equations of the Motion

The dynamical equation of a quantum system is governed by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} | \psi > = H | \psi > \qquad (4.2.1)$$

with the hermitean conjugate

$$-i\hbar < \psi | \frac{\partial}{\partial t} = < \psi | H , \qquad (4.2.2)$$

where the normalization of $| \psi >$ is time-independent

$$\frac{d}{dt} < \psi | \psi > = 0 . \qquad (4.3)$$

Heisenberg's quantization condition for the basic canonically conjugate operators P_x and X is

$$[P_x , X] = -i\hbar . \qquad (4.4)$$

For any observable A, the mean value over the normalized state $| \psi(t) >$ of a system is given by

$$< A > (t) = < \psi(t) | A | \psi(t) > , \qquad (4.5)$$

---

1 Some might think one could measure a complex observable by measuring separately its real part and pure imaginary parts, but this would involve two measurements or two observations. In general, it is not permissible to assume that two observations can be made exactly simultaneously, and if they can, they commute each other, and one thinks of them as two real dynamical variables.

which depends only on t. The evolution equation of the mean value of A is obtained by differentiating with respect to t

$$\frac{d}{dt} < A > (t) = \frac{1}{i\hbar} < [A, H] > + < \frac{\partial A}{\partial t} > . \qquad (4.6)$$

If A does not depend explicitly on time t, and commutes with H, then we say that A is a constant of the motion.

Within quantum theory, there are different representations or pictures in which the equations have different forms, but this familiar material will not be detailed further here.

4.3 The Uncertainty Principle

In the Pauli algebra, where all the numbers are real and the field $\mathcal{F}_{\mathcal{P}_0 \oplus \mathcal{P}_3}$ is composed of real scalar numbers and real pseudoscalars, how does one formulate the Uncertainty Principle?

In Stueckelberg's[19] real Hilbert space, in order to have the Uncertainty Principle, the extra operator J with $J^2 = -1$ had to be introduced. In the framework of Pauli algebra, although all numbers are real, the field $\mathcal{F}_{\mathcal{P}_0 \oplus \mathcal{P}_3}$ is said to be multi-vector valued (scalar and trivector) in $\mathcal{P}$, and the unit trivector "i" is very important for the realization of the Uncertainty Principle.

In $\mathcal{P}$, there are no complex numbers and therefore hermitean conjugation is just a reverse map (transpose) instead of a transpose plus a complex conjugate as in complex QM, and all the hermitean operators are symmetric linear operators in this real Hilbert space. For the operator A, which is hermitean,

$$A^* = A^t = A, \qquad (4.7)$$

where the superscript t is used for the transpose.

For two observables A and B, the criterion of impossibility of measuring A and B simultaneously is that A and B do not commute:

$$[A, B] \neq 0.$$

Following Stueckelberg's method, in real Hilbert space, it is easy to prove that: since $[A, B]$ is an antisymmetric operator with zero expectation value, $<|[A, B]|> = 0$, and the symmetric operator $[A, B]^2$ cannot lead to the Uncertainty Principle, there needs to be an antisymmetric operator J. In our formulation, all numbers are real, but the unit trivector "i" is an antisymmetric constant operator which changes sign under the reverse map $i^* = -i$. It can take the role of J, which therefore does not need to be introduced externally.

With the antisymmetric constant operator "i", one can thus construct a symmetric operator C:

$$C^* = C = i[A, B]. \tag{4.8}$$

One defines

$$\Delta A = A - < A >$$
$$\Delta B = B - < B >, \tag{4.9}$$

and introduces a function f of a real scalar λ

$$\begin{aligned} f(\lambda) &= |(\Delta A + \lambda i \Delta B)|\phi>|^2 \\ &= <\phi|(\Delta A - \lambda i \Delta B)(\Delta A + \lambda i \Delta B)|\phi> \\ &= <\Delta A^2> + \lambda^2 <\Delta B^2> + \lambda <|i[A, B]|> \geq 0, \end{aligned} \tag{4.10}$$

where $A^* = A, B^* = B$. The equation can be satisfied by a real scalar λ iff the discriminant of the quadratic polynomial $f(\lambda)$ is not positive:

$$<|i[A, B]|>^2 - 4 <\Delta A^2><\Delta B^2> \leq 0, \tag{4.11}$$

which means

$$< \Delta A^2 >< \Delta B^2 > \geq \frac{1}{4} < | i[A, B] | > . \qquad (4.12)$$

This is the familiar uncertainty relation. For example, if $A = P_x$, $B = X$, then

$$[P, X] = -i\hbar, \qquad (4.13)$$

thus in (4.8) $C = \hbar$, so that (4.12) implies

$$< \Delta P^2 >< \Delta X^2 > \geq \frac{1}{4}\hbar^2. \qquad (4.14)$$

4.4 The Time Reversal

In complex QM, the time-reversal transformation denoted by T is directly related to the imaginary number "i", and is effected by the combination of complex conjugation and the sign change of t. It is an antilinear but automorphic map, i.e., if $\alpha \in C$, where C is the complex field, and A and B are linear operators, then

$$(\alpha A)^T = T(\alpha A)T^{-1} = T\alpha T^{-1} T A T^{-1} = \alpha^* A^T, \qquad (4.15.1)$$

$$(AB)^T = T(AB)T^{-1} = T A T^{-1} T B T^{-1} = A^T B^T. \qquad (4.15.2)$$

However, in the present scheme, " + " takes the role of " * " in changing the sign of "i". Since " + " is an anti-automorphism, the realization of the time-reversal is very different from that in complex QM.

In order to construct the time reverse in QM over $\mathcal{F}_{p_0 \bullet p_3}$, one must first view the time-reversal operator in the ordinary complex QM[24].To obey the Correspondence Principle, the operators representing observables must transform under the time-reversal transformation in a same manner as in Classical Mechanics. This requires that

$$X' = TXT^{-1} = X, \tag{4.16.1}$$

$$P' = TPT^{-1} = -P, \tag{4.16.2}$$

$$L' = TLT^{-1} = -L. \tag{4.16.3}$$

With respect to the Schrödinger equation and the commutation relation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)> = H|\psi(t)>, \tag{4.17.1}$$

$$[P, X] = -i\hbar. \tag{4.17.2}$$

Since the time-reversal transformation is an automorphism (4.17.2) must be invariant. With eqs.(16.1; 16.2) the left-hand side of the commutation relation is transformed to

$$[P', X'] = T[P, X]T^{-1}$$

$$= [TPT^{-1}, TXT^{-1}] = -[P, X], \tag{4.18}$$

which leads to the right-hand side

$$TiT^{-1} = -i, \tag{4.19}$$

Therefore the time-reversal operator is an antilinear operator. In the Schrödinger equation, if $THT^{-1} = H$, i.e., the Hamiltonian is invariant under time-reversal, then

$$T\left(i\hbar\frac{\partial}{\partial t}|\psi(t)>\right) = Ti\hbar T^{-1}T\frac{\partial}{\partial t}T^{-1}T|\psi(t)> = -Ti\hbar T^{-1}\frac{\partial}{\partial t}T|\psi(t)>$$

$$= T(H|\psi(t)>) = THT^{-1}T|\psi(t)> = HT|\psi(t)>. \tag{4.20}$$

The relation $TiT^{-1} = -i$ makes the $|\psi(t)>, T|\psi(t)>$ satisfy the same Schrödinger equation, where the T acting on ψ gives $T\psi(t) = \psi^*(-t)$, first taking a complex conjugation of ψ and then changing the sign of t. The action of T can be written[24,26]

$$T = UK, \tag{4.21}$$

where U is a unitary operator and K is an antilinear operator whose application just gives the complex conjugate. For spinless particles, $T^2 = 1$ and

$$U^{-1} = U^* = U^\dagger. \tag{4.22}$$

Because T is an antilinear operator, for arbitrary states $|\psi>$, $|\phi>$ of a system

$$T(|\psi>) = |\psi'>, \tag{4.23.1}$$

$$T(|\phi>) = |\phi'>, \tag{4.23.2}$$

then we have the identity

$$<\phi'|\psi'> = <T\phi|T\psi> = <K\phi|K\psi> = = <\phi|\psi>^* = <\psi|\phi> \tag{4.24}$$

In real QM over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$, there are no complex numbers, and i is a pseudoscalar or a unit trivector. As indicated in the previous chapter, the antilinear transformations can be realized by two methods: one related to the map " + ", the other related to " ~ ". In the following, we will see that the former is related to time reversal, the latter to the spatial inversion. Hence the time-reversal operator T related to " + " should have the properties: if $\alpha \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ and A and B are linear operators, then

$$(\alpha A)^T = (A^T \alpha^T) = A^T \alpha^* = \alpha^* A^T, \tag{4.25.1}$$

$$(AB)^T = B^T A^T. \tag{4.25.2}$$

Now consider the state $|\psi(t)>$, which in the x-representation is

$$<x|\psi(t)> = \psi(t). \tag{4.26}$$

In complex quantum mechanics, the complex conjugate is defined by:

$$<x|\psi(t)>^* = <\psi(t)|x> = \psi^*(t). \tag{4.27.1}$$

In real QM on $\alpha \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$, the equivalent transformation is:

$$< x \mid \psi(t) >^{*} = < \psi(t) \mid x > = \psi^{*}(t), \qquad (4.27.2)$$

where ψ^{*} in complex QM and ψ^{*} in the present formalism result in the same sign change of i. In this sense, time-reversal has the same effect on the wave function ψ. In order that the time-reversal transformation denoted by the same T be written as a combination of an anti-automorphism denoted by K' (which takes the role of the complex conjugation) and a linear transformation denoted by U' (which only changes time t to -t) put[1]

$$T = K'U', \qquad (4.28)$$

$$K'K' = 1, \quad U'^{*} = U'^{-1}.$$

Here K' and U' are defined by

$$T(\psi(t)) = K'U'(\psi(t)) = K'(\psi(-t)) = \psi^{*}(-t), \qquad (4.29.1)$$

$$T(\mid \psi(t) >) = K'U'(\mid \psi(t) >) = K'(\mid U'\psi(t) >)$$

$$= < U'\psi(t) \mid = < \psi(-t) \mid, \qquad (4.29.2)$$

or defined within an inner product

$$< \phi(t) \mid \psi(t) >^{T} = < K'U'\phi(t) \mid K'U'\psi(t) > = < U'\phi(t) \mid U'\psi(t) >^{*}$$

$$= < U'\psi(t) \mid U'\phi(t) > = < \phi(t) \mid \psi(t) >^{*}. \qquad (4.30)$$

This is equivalent to eq.(4.24). For the property $T^2 = 1$ of spinless particles, it is easy to prove within an inner product

$$< \phi(t) \mid \psi(t) >^{T^2} = (< \phi(-t) \mid \psi(-t) >^{*})^{T} = < \phi(t) \mid \psi(t) >. \quad (4.31)$$

---

1 Or one might take $T = U''K'$. There is no really difference if $U'' = U''^{*}$.

In the following subsections, we will detail the time-reversal transformations on the Schrödinger equation and on operators P_x, X, L, and on the commutation relations, respectively.

4.4.1 The Time Reversal of the Schrödinger Equation

The Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)>=H|\psi(t)>, \qquad (4.32.1)$$

and its hermitean conjugate equation is

$$<\psi(t)|\frac{\partial}{\partial t}(-i\hbar)=<\psi(t)|H. \qquad (4.32.2)$$

Here the operator H is hermitean. In the x-representation, the equations are of the form

$$i\hbar\frac{\partial}{\partial t}\psi(t)=H\psi(t), \qquad (4.32.3)$$

$$(-i\hbar)\frac{\partial}{\partial t}\psi^*(t)=H\psi^*(t). \qquad (4.32.4)$$

When the time-reversal transformation T acts on the Schrödinger equation

$$T\left(i\hbar\frac{\partial}{\partial t}|\psi(t)>\right)=T(H|\psi(t)>), \qquad (4.33.1)$$

so that

$$T(|\psi(t)>)(i^T)\hbar\left(\frac{\partial}{\partial t}\right)^T=T(|\psi(t)>)H^T. \qquad (4.33.2)$$

If $H^T=H$, and since $\left(\frac{\partial}{\partial t}\right)^T=-\frac{\partial}{\partial t}$, one wants

$$T(|\psi(t)>)=K'|\psi(-t)>=<\psi(-t)|. \qquad (4.33.3)$$

As in (4.19), in order for $|\psi(t)>$ to satisfy the same Schrödinger equation, one must have

$$i^T = i^* = -i. \tag{4.34}$$

On other hand, if the x-representation is chosen, then the time-reversal transformation acting on the Schrödinger equation has the form

$$<x|i\hbar\frac{\partial}{\partial t}|\psi(t)>^T = <K'U'x|K'U'\left(i\hbar\frac{\partial}{\partial t}\right)|\psi(t)>$$

$$= <U'x|\left(i\hbar\frac{\partial}{\partial(-t)}\right)U|\psi(t)>^* = <x|\left(i\hbar\frac{\partial}{\partial(-t)}\right)|\psi(-t)>^* = i\hbar\frac{\partial}{\partial t}\psi^*(-t)$$

$$= <x|H|\psi(t)>^T = <K'U'x|K'U'H|\psi(t)>$$

$$= <x|H|\psi(-t)>^* = H\psi^*(-t). \tag{4.35}$$

Thus under such a definition of the time-reversal transformation, $\psi^*(-t) = T(\psi(t))$ and $\psi(t)$ satisfy the same Schrödinger equation as the $\psi^*(-t)$ and $\psi(t)$ in the ordinary complex QM.

4.4.2 The Time Reversal of the Operator X

The operator X in the x-representation acts on the $|\psi(t)>$ as defined by

$$X|\psi(t)> = \int dx X|x><x|\psi(t)> = x_c|\psi(t)>, \tag{4.36}$$

where the subscript " c " means that the quantity x_c is a classical number. The time-reversal transformation acts on this equation

$$<x|X|\psi(t)>^T = <x|U'XU'^{-1}U'|\psi(t)>^*$$

$$= <\psi(-t)|X^T|x> = X^T\psi^*(-t) = X^T T(\psi(t))$$

$$= x_c^T\psi^*(-t) = x_c\psi^*(-t) = XT(\psi(t)), \tag{4.37.1}$$

where

$$K'(U'XU'^{-1}) = X^T. \tag{4.38}$$

In the p-representation[1]

$$T(X \mid \psi(t)>) = T\Big(X \mid p> \int dp < p \mid \psi(t)>\Big)$$

$$= T\Big(\mid p> \int dp < p \mid \psi(t)>\Big)X^T = T\Big(i\hbar\frac{\partial}{\partial p} \mid p> \int dp < p \mid \psi(t)>\Big)$$

$$= < \psi(-t) \mid -p> \int dp < -p \mid \Big((-i\hbar)\frac{\partial}{\partial(-p)}\Big)$$

$$= < \psi(-t) \mid -p> \int dp < -p \mid X = < \psi(-t) \mid -p> \int dp < -p \mid (-X^*)$$

$$= XT(\mid \psi(t)>), \tag{4.37.2}$$

such that

$$T: \quad X - - - \rightarrow X' = X^T = X, \tag{4.39}$$

where both operators X^T and X act on the state $T(\psi(t))$.

This transformation rule for X in either representation is the same as eq.(4.16.1) in complex quantum mechanics.

4.4.3 The Time Reversal of the Operator P

For the operator P acting on the state $\mid \psi >$ in the x-representation :

$$< x \mid P \mid \psi(t)> = < x \mid \frac{\hbar}{i}\frac{\partial}{\partial x} \mid \psi(t)> = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(t). \tag{4.40}$$

Under the time-reversal transformation this becomes

---

1 Since $X^T = -X^* = X$, one must take care that X^* acting on the same state as X might have a different sign from the operator X in p-representation.

$$< x \mid P \mid \psi >^T = < U' x \mid U' P \psi(t) >^* = < U' P U'^{-1} U' \psi(t) \mid x >$$

$$= < \psi(-t) \mid P^T \mid x > = < x \mid U\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \mid \psi(t) >^*$$

$$= -\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^*(-t) = -P\psi^*(-t), \qquad (4.41.1)$$

where[1] $P^T = K(U' P U'^{-1}) = K(P) = P^*$. The eq(4.41.1) can be understood through the p-representation:

$$< x \mid P \mid \psi(t) >^T = \left(< x \mid P \mid p > \int dp < p \mid \psi(t) > \right)^T \qquad (4.41.2)$$

$$= \int dp < x \mid P \mid p >^T < p \mid \psi(t) >^T = \int dp < \psi(-t) \mid -p >< -p \mid P^T \mid x >$$

$$= \left(\int dp < x \mid p \mid p >< p \mid \psi(t) > \right)^T = \int dp < \psi(-t) \mid -p >< -p \mid -p \mid x >$$

$$= \int dp < \psi(-t) \mid -p >< -p \mid P^* \mid x > = < \psi(-t) \mid P^* \mid x > = -\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^*(-t),$$

where

$$\int dp \mid p >< p \mid = \int dp \mid -p >< -p \mid = 1$$

$$P \mid p > = p \mid p >, \quad P \mid -p > = -p \mid -p > . \qquad (4.42)$$

This is easily seen if one chooses the following simple wave function[2]

$$< x \mid \psi > = e^{i/\hbar p_c x}. \qquad (4.43)$$

---

1 One might think $P^T \stackrel{?}{=} P^* \stackrel{?}{=} P$, but here the interpretation is different. The definition of $P^* = P$ is that P acts on ψ while P^* acts on ψ^*. If one defines P and P^* acting on the same state, then he gets the result $P^* = -P$. Thus, here it should be understood that $P^T = P^* = -P$ are defined acting on the same state $T(\psi) = \psi^*(-t)$.

2 Here the normalization factor $(2\pi\hbar)^{-3/2}$ is omitted.

Then

$$< x \mid P \mid \psi > = p_c e^{\frac{i}{\hbar} p_c x} \tag{4.44}$$

and

$$\begin{aligned} & < x \mid P \mid \psi >^T = < \psi(-t) \mid P^T \mid x > \\ & = P^T \left(e^{\frac{i}{\hbar} p_c x} \right)^T = P^T e^{\frac{i}{\hbar} p_c x} \\ & = p_c^T \left(e^{\frac{i}{\hbar} p_c x} \right)^T = - p_c e^{\frac{i}{\hbar} p_c x} \\ & = - P e^{\frac{i}{\hbar} p_c x}. \end{aligned} \tag{4.45}$$

So, from eqs.(4.41.1; 4.41.2), we have

$$T: \quad P - - - \rightarrow P' = P^T = - P, \tag{4.46}$$

where both operators P and P^T operate on the same state $T(\psi(t))$.

This transformation rule for P in either representation is the same as eq.(4.16.2) in complex quantum mechanics.

4.4.4 The Time Reversal of the Operator L

L can be written as

$$L_i = \epsilon_{ijk} P_j X_k. \tag{4.47}$$

Under the time-reversal transformation, since

$$\forall j \neq k \ni [P_j, X_k] = 0 \tag{4.48}$$

the order of the components of P and X in L is unimportant and

$$L'_i = (L_i)^T = \epsilon_{ijk} X_k^T P_j^T = \epsilon_{ijk} P_j^T X_k^T = \epsilon_{ijk} X_k (- P_j) = - L_i. \tag{4.49}$$

This transformation rule for L is the same as eq.(4.16.3) in complex quantum mechanics.

4.4.5 The Interpretation of the Commutation Relation Under the Time Reversal

The commutation relation,

$$[P_x, X] = -i\hbar \neq 0, \tag{4.50}$$

becomes under the time-reversal transformation,

$$([P_x, X])^T = -[P_x^T, X^T] = [P_x, X] \stackrel{?}{=} (-i\hbar)^T = i\hbar \tag{4.51}$$

which seems a contradiction.

To solve this problem, first note that the time-reversal transformation is an anti-automorphism, which changes the order of the multiplied operators. Second, both the operator and the transformed operator act on the same state vector $T(\psi(t))$ and the interpretation of the transformation must depend on the state vector. This problem was seen to arise before, when one tried to interpret the hermitean operator P_x in the x-representation without regard to the space on which it acts. There, one questioned whether

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^* = -\frac{\hbar}{i}\frac{\partial}{\partial x} \stackrel{?}{=} \frac{\hbar}{i}\frac{\partial}{\partial x}. \tag{4.52}$$

The problem is resolved when it is understood that the operator P_x acts on a state $\psi(t)$, whereas the transformed operator P^* acts on the dual state $\psi^*(t)$. Thus both give identical results.

For the commutation relation, because time reversal is an anti-automorphism, one cannot consider it without the inner product or the state vectors on which the operator (or its transformed operator) acts. First let the commutator act on the state $|\psi>$,

$$[P_x, X]\,|\,\psi>, \tag{4.53}$$

and project it onto the $|\,x>$ representation:

$$<x\,|\,[P_x, X]\,|\,\psi>. \tag{4.54}$$

Under time reversal, in explicit form:

$$\begin{aligned}
&<x\,|\,[P_x, X]\,|\,\psi>^T = <\psi(-t)\,|\,(-[P_x^T, X^T]\,|\,x> \\
&= \psi^*(-t)(-1)\left(\left(-\frac{\hbar}{i}\right)\left(\frac{\partial}{\partial x}\right)x_c - x_c\left(-\frac{\hbar}{i}\right)\left(\frac{\partial}{\partial x}\right)\right) \\
&= (-1)\left(x_c\left(-\frac{\hbar}{i}\right)\left(\frac{\partial}{\partial x}\right) - \left(-\frac{\hbar}{i}\right)\left(\frac{\partial}{\partial x}\right)x_c\right)\psi^*(-t).
\end{aligned} \tag{4.55}$$

If one rewrites

$$\psi^*(-t) = \psi^T(t), \tag{4.56}$$

then

$$\begin{aligned}
&([P_x, X]\psi(t))^T = \psi^T(t)([P_x, X])^T \\
&= ([P_x^T, X^T])\psi^T(t) = -[P_x, X]\psi^T(t).
\end{aligned} \tag{4.57}$$

Here the most important point is that when the actions of operators are shifted from the left side to the right side, the order in which the operators act on the wave function must be maintained. This solves the problem.

The right hand side of the commutation relation acts on the state $|\,\psi>$, which when projected onto the $|\,x>$ representation gives

$$<x\,|\,-i\hbar\,|\,\psi> = -i\hbar\psi(t), \tag{4.58}$$

and under the time-reversal transformation:

$$\begin{aligned}
&<x\,|\,-i\hbar\,|\,\psi>^T = <\psi(-t)\,|\,(-i\hbar)^T\,|\,x> \\
&= (-1)(-i)\hbar <\psi(-t)\,|\,x> = (-1)(-i)\hbar\psi^*(-t) \\
&= (-1)(-i)\hbar\psi^T(t).
\end{aligned} \tag{4.59}$$

From another viewpoint, the above equation may be understood in the general form of an inner product of any $|\phi(t)>$ and $|\psi(t)>$. For the left hand side:

$$<\phi(t)|[P_x, X]|\psi(t)>^T = <K'\phi(-t)|K'([P_x, X]\psi(-t))>$$

$$= <\phi(-t)|[P_x, X]|\psi(-t)>^*, \qquad (4.60)$$

where the P_x and X are operators, not dependent on time explicitly[1], and

$$<\phi(-t)|[P_x, X]|\psi(-t)>^* = <([P_x, X])^*\phi(-t)|\psi(-t)>^* \qquad (4.61)$$

$$= <\psi(-t)|([P_x, X])^*|\phi(-t)> = (-1)<\psi(-t)|[P_x, X]|\phi(-t)>.$$

For the right hand side,

$$<\phi(t)|(-i\hbar)|\psi(t)>^T = <\phi(-t)|-i\hbar|\psi(-t)>^*$$

$$= (-1)<\psi(-t)|(-i\hbar)|\phi(-t)>. \qquad (4.62)$$

In this way, one obtains the result that the identity of the commutation relation of P_x and X is not destroyed by a time reversal transformation.

4.5 The Parity Transformation

As discussed in chapter 2, there are two types of transformation related to the parity transformation in classical mechanics:

1. Spatial inversion, which just changes the signs of the unit basis vectors in the frame transformations. It does not change the signs of the elements in $\mathcal{P}_0, \mathcal{P}_2$, but does change the signs of elements in $\mathcal{P}_1, \mathcal{P}_3$.
2. Spatial reversal, which is the combination of spatial inversion and the reverse transformation, i.e., it simultaneously changes the signs of the unit

1 This is similar to the calculation in complex QM when one tries to understand the time-reversal transformation as the product of complex conjugation and a change in the sign of the variable t.

vectors in the frame transformations and the order of the elements in the Pauli algebra. Thus it does not change the signs of the elements in $\mathcal{P}_0, \mathcal{P}_3$, but does change those of elements in $\mathcal{P}_1, \mathcal{P}_2$.

Now the problem arises: which one is a suitable transformation for the parity transformation in real Quantum Mechanics? In ordinary complex QM, the "i" is simply an imaginary number, whose sign is not changed by either the spatial inversion or the transpose transformations, but in the real Quantum Mechanics over the Pauli algebra, the situation changes. Here the "i" is a unit trivector or a pseudoscalar which changes sign under spatial inversion, but does not change sign under spatial reversal. In the next two subsections, these will be discussed separately.

4.5.1 The Spatial Inversion " ~ " Can Not Be a Suitable Parity Transformation

As is well known, any geometric symmetry operation can be performed in either the active (fixing axes, changing components) or passive (changing the axes) pictures. Thus, for any vector of the physical system with the general form

$$\vec{a} = a_i \sigma_i, \tag{4.63}$$

the spatial reflection can be achieved by two equivalent methods — by either a passive or an active transformation. The passive transformation changes the signs of the unit basis vectors or the basis axes

$$P(\vec{a}) = -\vec{a} = a_i P(\sigma_i) = a_i(-\sigma_i), \tag{4.64}$$

which can be rewritten

$$P(\vec{a}) = -\vec{a} = (-a_i)\sigma_i. \tag{4.65}$$

In real quantum mechanics, in order to convert the spatial inversion from a passive transformation to an active transformation, care must be exercised about one important aspect. From chapter 3, the Hilbert space is constructed over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$, and

all the linear operators (including those corresponding to vectors) acting on this space can only be in the component form of vector elements (or bivector elements[1]). Therefore, spatial inversion must be performed as an active transformation on the physical components of vectors rather than as a passive transformation on the unit basis vectors. However, the treatment of $"i"$ is different, as it is itself an element of the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$. It is a linear operator, and the state vectors in the Hilbert space are scalar- and trivector-valued functions. To date, there is no such direct classical physical quantity as the component of a pseudoscalar in QM. It changes its sign under an active spatial transformation as part of the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$. No corresponding procedure exists for $"i"$ when one changes from the passive transformation to an active transformation. Consequently, one must maintain the spatial transformation acting on $"i"$ when changing to an active transformation. Thus for spatial inversion in an active transformation[2], two points are assumed

1. All the component elements related to the vectors change their signs, where the component elements induced by vectors change their signs according to the inducing elements.
2. The unit trivector $"i"$ also changes its sign[3].

Under these assumptions, the spatial inversion has the same effects either in an active transformation or in a passive transformation.

The spatial inversion, denoted by $\hat{\Pi}$, is an automorphism. In order to construct the spatial inversion transformation $\hat{\Pi}$, since the sign of $"i"$ changes under either $\hat{\Pi}$ or " + ", I label it explicitly to trace the sign change.

---

1 The bivector can be induced by the bilinear form of two vectors.

2 It might better be called a "mixed transformation picture".

3 Some may question the point 2 and maintain that $"i"$ should not change its sign. If so, the $"i"$ will lose its meaning of being the pseudoscalar in the Pauli algebra.

Usually in complex quantum mechanics, one sets

$$|\psi>=\int d\vec{x}\,|\vec{x}><\vec{x}|\psi>=\int d\vec{x}\,\psi(x,y,z)\,|\vec{x}>, \qquad (4.66)$$

where $\psi(x,y,z)$ is a complex scalar-valued function. The parity operator denoted by Π is defined by the action on the basis vector[25] $|\vec{x}>$

$$\Pi\,|\vec{x}>=|-\vec{x}> \qquad (4.67)$$

and has no effect on $\psi(x,y,z)$

$$\Pi\,|\psi>=\Pi\int d\vec{x}\,\psi(x,y,z)\,|\vec{x}>=\int d\vec{x}\,\psi(x,y,z)\,|-\vec{x}>, \qquad (4.68)$$

so that

$$<\vec{x}\,|\,\Pi\,|\psi>=\psi(-x,-y,-z). \qquad (4.69)$$

In the Pauli algebra formulation, one can use notations to separate i explicitly:

$$|\psi>=|\psi;i>, \qquad (4.70.1)$$

$$<\vec{x}\,|\psi;i>=\psi(x,y,z;i), \qquad (4.70.2)$$

$$<\vec{p}\,|\psi;i>=\psi(p_x,p_y,p_z;i), \qquad (4.70.3)$$

$$<\vec{x}\,|\psi;i>^{*}=\psi^{*}(x,y,z;i)$$

$$=\psi(x,y,z;-i)=<\psi;i\,|\vec{x}>, \qquad (4.70.4)$$

$$<\vec{p}\,|\psi;i>^{*}=\psi^{*}(p_x,p_y,p_z;i)$$

$$=\psi(p_x,p_y,p_z;-i)=<\psi;i\,|\vec{p}>. \qquad (4.70.5)$$

Other notations and relations are the same as in complex QM. With this notation

$$|\psi;i>=\int d\vec{x}\,|\vec{x}><\vec{x}|\psi;i>=\int d\vec{x}\,\psi(x,y,z;i)\,|\vec{x}>. \qquad (4.71)$$

Here one must note that $<\vec{x} \mid \psi; i>$ is a real scalar- and trivector-valued function, so that when $\hat{\Pi}$ acts on the basis vectors, i.e., a passive transformation is performed, the i will change its sign:

$$\hat{\Pi}\psi(x, y, z; i) = \hat{\Pi} <\vec{x} \mid \psi; i> = <\vec{x} \mid \psi; -i> = \psi(x, y, z; -i) \quad (4.72)$$

whence

$$\hat{\Pi} \mid \psi; i> = \hat{\Pi}\left(\int d\vec{x} \mid \vec{x}><\vec{x} \mid \psi; i>\right) = \int d\vec{x} \mid -\vec{x}><\vec{x} \mid \psi; -i> \quad (4.73)$$

which when projected onto $\mid \vec{x}>$ gives

$$<\vec{x} \mid \hat{\Pi} \mid \psi; i> = \psi(-x, -y, -z; -i). \quad (4.74)$$

These results illustrate the assumptions that in changing from a passive spatial transformation to an active transformation one must maintain the change in the sign of i as well as change the signs of components of any vectors. Thus the passive or active transformations of spatial inversion have the same results.

From the above discussions, we can prove the following properties of $\hat{\Pi}$:

1. $\hat{\Pi}$ is a automorphism with the property $\hat{\Pi}^2 = 1$;

2. $\hat{\Pi}$ is an antilinear operator;

3. $\hat{\Pi}$ is an anti-unitary operator instead of an unitary operator.

For the property (1), using eq(4.73), one has

$$\hat{\Pi}\hat{\Pi} \mid \psi; i> = \hat{\Pi} \int d\vec{x} \mid -\vec{x}><\vec{x} \mid \psi; -i>$$

$$= \int d\vec{x} \mid \vec{x}><\vec{x} \mid \psi; i> = \mid \psi; i> . \quad (4.75)$$

For the property (2), if $\exists \mid \psi_1; i>$, $\mid \psi_2; i>$ and $\mid \psi; i>$ is

$$\mid \psi; i> = \alpha_1 \mid \psi_1; i> + \alpha_2 \mid \psi_2; i>, \quad (4.76)$$

where a_1, $a_2 \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$, then

$$\tilde{\Pi} \mid \psi; i> = a_1^* \tilde{\Pi} \mid \psi_1; i> + a_2^* \tilde{\Pi} \mid \psi_2; i> . \tag{4.77}$$

For property (3), if one considers only spinless particles, because of

$$< \vec{x} \mid \psi; -i> = \psi(x, y, z; -i) = \psi^*(x, y, z; i) = < \psi; i \mid \vec{x} >, \tag{4.78}$$

one can make the symbolic relation

$$\mid \psi; -i> = < \psi; i \mid \tag{4.79}$$

Now if there are any two state vectors $\mid \phi; i>$, $\mid \psi; i>$, such that

$$\mid \phi'; i> = \tilde{\Pi} \mid \phi; i>, \quad \mid \psi'; i> = \tilde{\Pi} \mid \psi; i>, \tag{4.80}$$

then

$$\begin{aligned} < \phi'; i \mid \psi'; i> &= (< \phi; i \mid \tilde{\Pi}^*)(\tilde{\Pi} \mid \psi; i>) \\ &= \int d\vec{x} \int d\vec{x}' < \phi; -i \mid x'>< -\vec{x}' \mid -\vec{x}>< \vec{x} \mid \psi; -i> \\ &= \int d\vec{x} < \phi; -i \mid \vec{x}>< \vec{x} \mid \psi; -i> = < \phi; -i \mid \psi; -i> \\ &= < \phi; i \mid \psi; i>^* = < \psi; i \mid \phi; i>, \end{aligned} \tag{4.81}$$

thus $\tilde{\Pi}$ is an anti-unitary operator, instead of the unitary operator Π in complex QM for which

$$< \phi \mid \Pi^* \Pi \mid \psi > = < \phi \mid \psi > . \tag{4.82}$$

In the following, it will be shown that spatial inversion with the above three properties is not a suitable parity transformation in real QM with transformation rules consistent with those in complex QM or in classical physics. For example, simply by applying $\tilde{\Pi}$ to the commutation relation of operators P_x and X, an insurmountable difficulty arises. Because $\tilde{\Pi}$ is an antilinear automorphic operator,

$$\hat{\Pi}[P_x, X]\hat{\Pi}^{-1} = [\hat{\Pi}P_x\hat{\Pi}^{-1}, \hat{\Pi}X\hat{\Pi}^{-1}]$$

$$\stackrel{?}{=} \hat{\Pi}i\hbar\hat{\Pi}^{-1} = -i\hbar. \qquad (4.83)$$

It is easy to see that the transformation properties between P_x and X must not be the same, for otherwise there will be a contradiction.

For example, if we take the simple state function describing some physical system as before:

$$< x \mid \psi > = e^{\frac{i}{\hbar}p_x x}, \qquad (4.84)$$

then the operator $\hat{\Pi}$ acts on it

$$\hat{\Pi}\psi(x) = e^{-\frac{i}{\hbar}p_x x}, \qquad (4.85)$$

where the "i" as well as the classical quantities p_x and x all change signs under the action of $\hat{\Pi}$. Now taking the operators P_x and X into consideration:

$$X\psi(x) = x\psi(x), \qquad (4.86.1)$$

$$P_x\psi(x) = \left(-i\frac{\partial}{\partial x}\right)\psi(x) = p_x\psi(x). \qquad (4.86.2)$$

Under the spatial inversion $\hat{\Pi}$,

$$\begin{aligned}\hat{\Pi}(X\psi(x)) &= \hat{\Pi}X\hat{\Pi}^{-1}\hat{\Pi}(\psi(x)) \\ &= \hat{\Pi}(x\psi(x)) = \hat{\Pi}x\hat{\Pi}^{-1}\hat{\Pi}(\psi(x)) \\ &= -x\hat{\Pi}(\psi(x)) = (-X)\hat{\Pi}(\psi(x)), \qquad (4.87.1)\end{aligned}$$

$$\tilde{\Pi}(P_x\psi(x)) = \tilde{\Pi}P_x\tilde{\Pi}^{-1}\tilde{\Pi}(\psi(x))$$

$$= \tilde{\Pi}\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\tilde{\Pi}^{-1}\tilde{\Pi}(\psi(x)) = \tilde{\Pi}p_x\tilde{\Pi}^{-1}\tilde{\Pi}(\psi(x))$$

$$= -p_x\tilde{\Pi}(\psi(x)) = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)(e^{-ip_x x})$$

$$= P_x\tilde{\Pi}(\psi(x)), \qquad (4.87.2)$$

so that

$$\tilde{\Pi}: \quad X \dashrightarrow X' = \tilde{\Pi}X\tilde{\Pi}^{-1} = -X, \qquad (4.88.1)$$

$$\tilde{\Pi}: \quad P_x \dashrightarrow P'_x = \tilde{\Pi}P_x\tilde{\Pi}^{-1} = P_x \qquad (4.88.2)$$

which does not make sense and is not consistent with the parity transformation in classical physics which is denoted by P

$$P: \quad x_c \dashrightarrow x_c' = Px_cP^{-1} = -x_c, \qquad (4.88.3)$$

$$P: \quad p_{xc} \dashrightarrow p'_{xc} = Pp_{xc}P^{-1} = -p_{xc} \qquad (4.88.4)$$

and is not consistent with that in complex quantum mechanics

$$\Pi: \quad X \dashrightarrow X' = \Pi X\Pi^{-1} = -X, \qquad (4.88.5)$$

$$\Pi: \quad P_x \dashrightarrow P'_x = \Pi P_x\Pi^{-1} = -P_x. \qquad (4.88.6)$$

If one takes the p-representation, he will get the transformation rules of P_x changing the sign, but X not changing the sign. The transformation rules of X and P_x are so complicated: in one way they are transformed like this, in another way, transformed very differently. Thus I choose the spatial reversal transformation for purposes of constructing the parity transformation in real quantum mechanics.

4.5.2 The Spatial Reversal

The above subsection showed that the spatial inversion transformation $\tilde{\Pi}$ as an antilinear, anti-unitary operator is not a suitable transformation for the parity operator in real QM. The following will show how to construct the parity transformation.

The Pauli-algebra approach to special relativity[14],gives a hint that the spatial reversal " - " might be more suitable for the parity operator when using real Pauli algebra as a tool. In Ref.[14], in order to obtain the Minkowski metric, the spatial reversal instead of spatial inversion is used. This was done even though in classical physics, where there are no basic physical quantities described directly by bivectors or trivectors $"i"$, the spatial inversion and the spatial reversal are identical[1].

For example, if instead of the spatial reversal, the spatial inversion " ~ " is used to construct the scalar from the 4-vector a

$$a = a_0 + \vec{a}, \tag{4.89}$$

then since the spatial inversion of a is

$$\tilde{a} = \tilde{\Pi} a \tilde{\Pi}^{-1} = a_0 - \vec{a}, \tag{4.90}$$

the product is

$$a\tilde{a} = \tilde{a}a = a_0^2 - \vec{a}^2, \tag{4.91}$$

which is a Lorentz scalar. There is no difference between these two transformations. However, if a is a general element in the Pauli algebra

$$a = a_0 + \vec{a}_1 + i\vec{a}_2 + ia_3 = e^{b_0 + \vec{w} + i\vec{\theta} + iv}, \tag{4.92}$$

---

1 In the Pauli algebra approach to generalized electromagnetic theory with monopoles, there is a difference between the two transformations.

then neither $\alpha\tilde{\alpha}$ nor $\tilde{\alpha}\alpha$ is a Lorentz scalar (a scalar and a pseudoscalar in the Pauli algebra), but $\alpha\bar{\alpha}$ or $\alpha\bar{\alpha}$ is. Thus in this respect, the spatial reversal transformation is more general than the spatial inversion transformation.

On the other hand, the spatial reversal is the combination of the transpose " + " and the spatial inversion " ~ " so that

$$\bar{i} = \tilde{i}^* = [(-\sigma_1)(-\sigma_2)(-\sigma_3)]^* = -i^* = i, \tag{4.93}$$

and thus the "i" in the Pauli algebra does not change sign under the spatial reversal. This is more consistent with the behaviour in complex quantum mechanics where the "i" is unchanged. The next step is therefore to construct the spatial reversal as a parity transformation in real quantum mechanics.

Since, in the Pauli algebra, the map " - " is a combination of " ~ " and " + ", it can be written as

$$\bar{\Pi} = K'\tilde{\Pi}, \tag{4.94}$$

where $\tilde{\Pi}$ was defined before as a spatial inversion, an antilinear automorphism, while K' has the meaning " + ". For example, for $\forall\psi(x, y, z; i)$

$$K'\psi(x, y, z; i) = \psi^*(x, y, z; i). \tag{4.95}$$

Let us use the notation

$$K' \mid \psi; i > = < \psi; i \mid, \tag{4.96}$$

which when projected onto $\mid \vec{x} >$ gives

$$(K' \mid \psi; i >) \mid \vec{x} > = < \psi; i \mid \vec{x} > = \psi^*(x, y, z; i). \tag{4.97}$$

For all operators A, B,

$$K'(AB) = K'(B)K'(A). \tag{4.98}$$

With this definition, the transformation rules of $\overline{\Pi}$ acting on $|\psi;i>$ are readily found from the above, namely

$$\overline{\Pi}\,|\psi;i>=K'\tilde{\Pi}\,|\psi;i>, \tag{4.99}$$

with eq(4.73).

$$\overline{\Pi}\,|\psi;i>=K'\int d\vec{x}'\,|-\vec{x}'><\vec{x}'\,|\psi;-i>$$

$$=\int d\vec{x}'<\psi;-i\,|\vec{x}'><-\vec{x}'\,|, \tag{4.100.1}$$

$$\overline{\Pi}\,|\psi;i>=K'\int d\vec{p}'\,|-\vec{p}'><\vec{p}'\,|\psi;-i>$$

$$=\int d\vec{p}'<\psi;-i\,|\vec{p}'><-\vec{p}'\,|, \tag{4.100.2}$$

projecting onto $|\vec{x}>$ or $|\vec{p}>$

$$(\overline{\Pi}\,|\psi;i>)\,|\vec{x}>=<\psi;-i\,|-\vec{x}>=\psi^*(-x,-y,-z;-i) \tag{4.101.1}$$

$$(\overline{\Pi}\,|\psi;i>)\,|\vec{p}>=<\psi;-i\,|-\vec{p}>=\psi^*(-p_x,-p_y,-p_z;-i) \tag{4.101.2}$$

From here, it is easy to prove the following properties of $\overline{\Pi}$:

1. $\overline{\Pi}$ is an anti-automorphism with property $\overline{\Pi}^2=1$;

2. $\overline{\Pi}$ is a linear operator;

3. $\overline{\Pi}$ is a unitary operator.

For the property (1), using eq(4.100.1) and symbolic notation eq(4.79), one has

$$\overline{\Pi}\,\overline{\Pi}\,|\psi;i>=\overline{\Pi}\,|\psi(-x,-y,-z;-i)>=|\psi(x,y,z;i)>. \tag{4.102}$$

For the property (2), it is self-evident that any product of two antilinear operator is a linear operator. For $|\psi_1;i>$, $|\psi_2;i>$ and $|\psi;i>$ defined by

$$|\psi;i>=\alpha_1\,|\psi_1;i>+\alpha_2\,|\psi_2;i>, \tag{4.103}$$

where α_1, $\alpha_2 \in \mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$, then

$$\overline{\Pi} \mid \psi; i > = K'(\alpha_1^* \tilde{\Pi} \mid \psi_1; i > + \alpha_2^* \tilde{\Pi} \mid \psi_2; i >)$$

$$= \alpha_1 \overline{\Pi} \mid \psi_1; i > + \alpha_2 \overline{\Pi} \mid \psi_2; i > \qquad (4.104)$$

From the property (3), if there are any two state vectors $\mid \phi; i >$, $\mid \psi; i >$, such that

$$\mid \phi'; i > = \overline{\Pi} \mid \phi; i >, \quad \mid \psi'; i > = \overline{\Pi} \mid \psi; i >, \qquad (4.105)$$

then

$$< \phi'; i \mid \psi'; i > = (< \phi; i \mid \overline{\Pi}^*)(\overline{\Pi} \mid \psi; i >)$$

$$= \{(< \phi; i \mid \tilde{\Pi}^*)(\tilde{\Pi} \mid \psi; - >)\}^* = \{< \phi; - i \mid \psi; - i >\}^*$$

$$= \{< \phi; i \mid \psi; i >^*\}^* = < \phi; i \mid \psi; i > . \qquad (4.106)$$

For spinless particles, the three transformations[1] Π, $\tilde{\Pi}$ and $\overline{\Pi}$ acting on a state $< \vec{x}; i \mid \psi > = \psi(x, y, z; i)$ have the relationships:

$$\tilde{\Pi} \psi(x, y, z; i) = \psi(-\vec{x}; - i) = \psi^*(-\vec{x}; i) = \Pi \psi^*(\vec{x}; i), \qquad (4.107.1)$$

$$\overline{\Pi} \psi(x, y, z; i) = K'(\tilde{\Pi}(\psi(x, y, z; i)))$$

$$= K' \psi^*(-\vec{x}; i) = \psi(-\vec{x}; i) = \Pi \psi(\vec{x}; i), \qquad (4.107.2)$$

and on $< \vec{p}; i \mid \psi > = \psi(\vec{p}; i)$

$$\tilde{\Pi} \psi(\vec{p}; i) = \psi(-\vec{p}; - i) = \psi^*(-\vec{p}; i) = \Pi \psi^*(\vec{p}; i), \qquad (4.107.3)$$

$$\overline{\Pi} \psi(\vec{p}; i) = K'(\tilde{\Pi}(\psi(\vec{p}; i)))$$

$$= K' \psi^*(-\vec{p}; i) = \psi(-\vec{p}; i) = \Pi \psi(\vec{p}; i), \qquad (4.107.4)$$

---

1 The operator Π is not an operator in real QM, but it is assumed be the same as that in complex QM.

where the most important thing is to note that the result of $\overline{\Pi}$ acting on ψ which is valued over the field of $\mathcal{F}_{\mathcal{P}_0 \bullet \mathcal{P}_3}$ is equal to that of Π acting on ψ which is valued over the field of complex numbers.

Below, I will consider the spatial-reversal transformations on the Schrödinger equation, on the operators P_x, X, L, and on the commutation relations.

4.5.3 The Spatial Reversal of the Schrödinger Equation

The Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} | \psi; i > = H | \psi; i >, \qquad (4.108)$$

under the action of the spatial reversal $\overline{\Pi}$ becomes

$$\overline{\Pi}\left(i\hbar \frac{\partial}{\partial t} | \psi; i > \right) = \overline{\Pi}(| \psi; i >) \overline{\Pi}\left(i\hbar \frac{\partial}{\partial t} \right) = (i\hbar \frac{\partial}{\partial t} \overline{\Pi}(| \psi; i >)$$

$$= \overline{\Pi}(H | \psi; i >) = \overline{\Pi}(| \psi; i >) \overline{\Pi}(H), \qquad (4.109)$$

since $H^* = H$, and if H has even parity, i.e., $\overline{\Pi}(H) = \tilde{\Pi}(H) = II$, then

$$i\hbar \frac{\partial}{\partial t} \overline{\Pi}(| \psi; i >) = H \overline{\Pi}(| \psi; i >). \qquad (4.110)$$

This means that $| \psi; i >$ and $\overline{\Pi}(| \psi; i >)$ satisfy the same Schrödinger equation. Eq(4.101.1) can be used to project onto $| \vec{x} >$

$$i\hbar \frac{\partial}{\partial t} \psi^*(-\vec{x}; -i) = H \psi^*(-\vec{x}; -i), \qquad (4.111.1)$$

or

$$i\hbar \frac{\partial}{\partial t} \psi(-\vec{x}; i) = H \psi(-\vec{x}; i), \qquad (4.111.2)$$

which is the same as that in complex QM.

4.5.4 The Spatial Reversal of Operator X

The operator X, in the $|\vec{x}>$ representation, acts on the $|\psi;i>$ according to:

$$X|\psi;i> = \int d\vec{x}\, x_c |\vec{x}><\vec{x}|\psi;i>, \qquad (4.112)$$

where the subscript " c " means that the quantity x_c is a classical number. Before making a spatial-reversal transformation, note that if

$$X|\vec{x}> = x_c|\vec{x}>, \qquad (4.113)$$

then

$$X|-\vec{x}> = X|\vec{x}'> \int d^3\vec{x}' <\vec{x}'|-\vec{x}>$$

$$= \int d^3\vec{x}' x'_c |\vec{x}'> \delta(-\vec{x}-\vec{x}') = -x_c|-\vec{x}>. \qquad (4.114)$$

Thus under the spatial-reversal transformation

$$\overline{\Pi}(X|\psi;i>) = \overline{\Pi}(X)\overline{\Pi}|\psi;i> = x_c <\psi;-i|\vec{x}> \int d\vec{x} <-\vec{x}| = <\psi;-i|\vec{x}> \int d\vec{x} <-\vec{x}|x_c$$

$$= <\psi;-i|\vec{x}> \int d\vec{x} <-\vec{x}|(-X^*) = -X\overline{\Pi}(|\psi;i>), \qquad (4.115)$$

where eq(4.101.1) was used[1], such that

$$\overline{\Pi}: \quad X \dashrightarrow X' = \overline{\Pi}(X) = X^{\overline{\Pi}} = -X, \qquad (4.116)$$

Also X and $X^{\overline{\Pi}}$ are both supposed to act on the same state $\overline{\Pi}|\psi;i>$.

This transformation rule for X is the same as eq.(4.88.5) in complex quantum mechanics.

---

1 Here the action of $\overline{\Pi}$ is passive, only acting on the basis vectors and not on the component x_c.

4.5.5 The Spatial Reversal of Operator P_x

For the operator P_x acting on $|\psi;i>$, one can expand in the $|\vec{p}>$ representation:

$$P_x|\psi;i> = P_x|\vec{p}>\int d\vec{p}<\vec{p}|\psi;i> = p_x|\vec{p}>\int d\vec{p}<\vec{p}|\psi;i> \quad (4.117)$$

Thus, under the spatial reversal transformation $\overline{\Pi}$

$$\overline{\Pi}(P_x|\psi;i> = \overline{\Pi}(P_x)\overline{\Pi}(|\psi;i>)$$

$$= \overline{\Pi}\left(p_x|p>\int d\vec{p}<p|\psi;i>\right) = <\psi;-i|\vec{p}>\int d\vec{p}<-\vec{p}|p_x$$

$$= <\psi;-i|\vec{p}>\int d\vec{p}<-\vec{p}|(-1)P_x = (-P_x)\overline{\Pi}(|\psi;i>), \quad (4.118)$$

where eq(4.100.2) was used and $\overline{\Pi}$ acts only on the basis vectors, i.e., the transformation is passive. In the $|\vec{x}>$ representation, since

$$P_x|\psi;i> = P_x|\vec{x}'>\int d\vec{x}'<\vec{x}'|\psi;i>$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial x'}|\vec{x}'>\int d\vec{x}'<\vec{x}'|\psi;i>, \quad (4.119)$$

the spatial-reversal transformation $\overline{\Pi}$ acting on the basis vectors gives

$$\overline{\Pi}(P_x|\psi;i> = \int d\vec{x}'<\vec{x}'|\psi;-i>^*<-\vec{x}'|\frac{\hbar}{i}\left(\frac{\partial}{\partial x'}\right). \quad (4.120)$$

When this is projected onto $|\vec{x}>$, since

$$\int d\vec{x}' < \vec{x}' \mid \psi; -i >^* < -\vec{x}' \mid \frac{\hbar}{i}\frac{\partial}{\partial x'} \mid \vec{x} >$$

$$= \int d\vec{x}' \psi^*(x', y', z'; -i)\left(\frac{\hbar}{i}\frac{\partial}{\partial x'}\right)\delta(\vec{x} + \vec{x}') = \psi^*(-x, -y, -z; -i)\frac{\hbar}{i}\left(-\frac{\partial}{\partial x}\right)$$

$$= \left(-\frac{\hbar}{i}\frac{\partial}{\partial x'}\right)\psi^*(-x, -y, -z; -i), \qquad (4.121)$$

then

$$\overline{\Pi}(P_x)\psi^*(-x, -y, -z; -i) = \overline{\Pi}(P_x)\overline{\Pi}(\psi(x, y, z; i))$$

$$= -\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^*(-x, -y, -z; -i) = -P_x\psi^*(-x, -y, -z; -i)$$

$$= -P_x\overline{\Pi}(\psi(x, y, z; i)). \qquad (4.122)$$

Thus in either representation

$$\overline{\Pi}: \quad P_x - - - \rightarrow P_x' = \overline{\Pi}(P_x) = P_x^{\overline{\Pi}} = -P_x. \qquad (4.123)$$

This transformation rule for P_x is the same as eq.(4.88.6) in complex quantum mechanics.

4.5.6 The Spatial Reversal of Operator L

L can be written as

$$L_i = \epsilon_{ijk} P_j X_k. \qquad (4.124)$$

Since

$$\forall j \neq k \ni [P_j, X_k] = 0, \qquad (4.125)$$

the orders of the components of P_x and X in L is unimportant. Under spatial reversal

$$L'_i = (L_i)^{\overline{\Pi}} = \epsilon_{ijk} X_k^{\overline{\Pi}} P_j^{\overline{\Pi}} = \epsilon_{ijk} P_j^{\overline{\Pi}} X_k^{\overline{\Pi}} = \epsilon_{ijk}(-X_k)(-P_j) = L_i. \qquad (4.126)$$

This transformation rule for L is the same as that in complex quantum mechanics.

4.5.7 The Interpretation of the Commutation Relation Under Spatial Reversal

The commutation relation

$$[P_x, X] = -i\hbar \neq 0, \tag{4.127}$$

leads to a problem because under the spatial-reversal transformation $\overline{\Pi}$:

$$\begin{aligned}([P_x, X])^{\overline{\Pi}} &= -[P_x^{\overline{\Pi}}, X^{\overline{\Pi}}] = -[P_x, X] \\ &\overset{?}{=} (i\hbar)^{\overline{\Pi}} = i\hbar.\end{aligned} \tag{4.128}$$

Again this gives a contradiction. The problem comes from the fact that the spatial-reversal transformation $\overline{\Pi}$, like the time reversal $\mathcal{T}$, is an anti-automorphism; its actions are meaningful only within inner products or on the state vectors.

Now the spatial-reversal transformation $\overline{\Pi}$ acts on the left-hand side of the general inner product for any two $|\phi(\vec{x};i)>$ and $|\psi(\vec{x};i)>$, according to

$$\begin{aligned}<\phi(\vec{x};i)|[P_x, X]|\psi(\vec{x};i)>^{\overline{\Pi}} &= <K'\tilde{\Pi}\phi(\vec{x};i)|K'\tilde{\Pi}([P_x, X]\psi(\vec{x};i))> \\ &= <\tilde{\Pi}\phi(\vec{x};i)|\tilde{\Pi}[P_x, X]\tilde{\Pi}^{-1}\tilde{\Pi}|\psi(\vec{x};i)>^* \\ &= (-1)<\tilde{\Pi}\phi(\vec{x};i)|[P_x, X]\tilde{\Pi}|\psi(\vec{x};i)>^*.\end{aligned} \tag{4.129}$$

where the identity

$$\tilde{\Pi}[P_x, X]\tilde{\Pi}^{-1} = (-1)[P_x, X] = i\hbar \tag{4.130}$$

was used. For the right-hand side of the commutation relation,

$$\begin{aligned}<\phi(\vec{x};i)|-i\hbar|\psi(\vec{x};i)>^{\overline{\Pi}} &= <K'\tilde{\Pi}\phi(\vec{x};i)|K'\tilde{\Pi}(-i\hbar\psi(\vec{x};i))> \\ &= <\tilde{\Pi}\phi(\vec{x};i)|\tilde{\Pi}-i\hbar\tilde{\Pi}^{-1}\tilde{\Pi}|\psi(\vec{x};i)>^* \\ &= (-1)<\tilde{\Pi}\phi(\vec{x};i)|(-i\hbar)\tilde{\Pi}|\psi(\vec{x};i)>^*.\end{aligned} \tag{4.131}$$

The result is thus obtained that the identity of the commutation relation of P_x and X is not destroyed by the spatial-reversal transformation.

CHAPTER 5. APPLICATIONS

5.1 The Superselection Rule From the Parity Transformation

5.1.1 The Even and Odd States

In section 4.5.2, the spatial reversal transformation $\overline{\Pi}$ with the property $\overline{\Pi}^2 = 1$ was defined so that its action on a state vector $\psi(\vec{x};t)$ gives

$$\overline{\Pi}\psi(\vec{x};t) = \psi^*(-\vec{x};-t), \tag{5.1}$$

which is symbolically expressed by

$$\overline{\Pi}(|\psi(\vec{x};t)>=|\psi^*(-\vec{x};-t)>=<\psi(-\vec{x};-t)|. \tag{5.2}$$

Now, $\overline{\Pi}$ can be used to define projection operators $\overline{P}_+$, $\overline{P}_-$

$$\overline{P}_\pm = \frac{1}{2}(1 \pm \overline{\Pi}). \tag{5.3}$$

It is easy to check that

$$\overline{P}_\pm^2 = \overline{P}_\pm, \tag{5.4.1}$$

$$\overline{P}_+\overline{P}_- = \overline{P}_-\overline{P}_+ = 0, \tag{5.4.2}$$

$$\overline{P}_+ + \overline{P}_- = 1. \tag{5.4.3}$$

Therefore, for a space $\mathcal{E}$, $\forall\psi \in \mathcal{E}$ can be decomposed into

$$|\psi> = (\overline{P}_+ + \overline{P}_-)|\psi> = |\psi_+> + |\psi_->, \tag{5.5}$$

where

$$|\psi_\pm> = \overline{P}_\pm|\psi(\vec{x};t)> = \frac{1}{2}(|\psi(\vec{x};t)> \pm |\psi^*(-\vec{x};-t)>), \tag{5.6}$$

has positive or negative parity

$$\overline{\Pi}|\psi_\pm> = \pm|\psi_\pm>. \tag{5.7}$$

$$\overline{\Pi} \mid \psi_{\pm} > = \pm \mid \psi_{\pm} > . \qquad (5.7)$$

In this way, $\mathcal{E}$ is decomposed into two orthogonal subspaces $\mathcal{E}_+$ and $\mathcal{E}_-$.

5.1.2 The Selection Rules of the Parity

For an arbitrary operator A, the transformed operator under $\overline{\Pi}$ is defined by

$$A^{\overline{\Pi}} = \overline{\Pi}(A). \qquad (5.8)$$

If $A^{\overline{\Pi}} = \eta A$, when $\eta = +1$, the operator A is said to be an even operator; when $\eta = -1$, it is an odd operator.

Let us calculate the matrix element of A over the states $\mid \psi_{\eta_\psi} >$ and $\mid \phi_{\eta_\phi} >$, where subscripts $\eta_{\psi(\phi)}$ mean that the two states have either even or odd parities:

$$< \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} > .$$

Under the spatial reversal

$$< \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} >^{\overline{\Pi}} = \eta_\phi \eta \eta_\psi < \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} >, \qquad (5.9)$$

but also from eq(4.106),

$$< \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} >^{\overline{\Pi}} = < \overline{\Pi} \phi_{\eta_\phi} \mid \overline{\Pi}(A_\eta \psi_{\eta_\psi}) > = < \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} > . \qquad (5.10)$$

Therefore

$$< \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} >^{\overline{\Pi}} = < \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} > = \eta_\phi \eta \eta_\psi < \phi_{\eta_\phi} \mid A_\eta \mid \psi_{\eta_\psi} >, \qquad (5.11)$$

which gives the selection rules:

1. If A_η is an odd-parity operator, then its matrix elements are zero between two states of even parity, or between two states of odd parity.

In particular, when $A = 1$, the (interaction-free) transition probability between two states of opposite parity is identically equal to zero.

5.2 Is $\overline{\Pi}$ a Constant of Motion?

If $\overline{\Pi}(H) = H^{\overline{\pi}} = H$, and if $|\psi>$ is a solution of the Schrödinger equation, then the projections $|\psi_{\pm}>$ obey

$$i\hbar\frac{\partial}{\partial t}|\psi_{\pm}> = H|\psi_{\pm}>, \tag{5.12}$$

and are thus also solutions of the system with definite even or odd parities, and the parity of the system is conserved. The following shows that the operator $\overline{\Pi}$ is then a constant of motion, as is Π[26]in complex QM.

In complex QM, an observable A is a constant of motion if it does not depend explicitly on time and if it commutes with H:

$$\frac{\partial A}{\partial t} = 0$$

$$[A, H] = 0. \tag{5.13}$$

According to this, an observable operator Π in complex QM is a constant of motion if $[\Pi, H] = 0$.

For the real formulation of QM, $\overline{\Pi}$ was seen in the previous chapter to be a suitable operator for the parity transformation. Although it is a linear operator, it is an anti-automorphism and hence is different from the Π of a linear automorphism in complex QM. How is one to interpret the conditions for $\overline{\Pi}$ to be a constant of motion?

First, from chapter 4, since $\overline{\Pi}^2 = 1$ and $\overline{\Pi}^{\dagger}\overline{\Pi} = 1$,

$$\overline{\Pi} = \overline{\Pi}^{*} \tag{5.14}$$

is a hermitean operator, and thus like Π in complex QM is an observable.

Second, the mean value of $\overline{\Pi}$ over the normalized state $|\psi(t)>$ of a system

$$<\overline{\Pi}>(t) = <\psi(t)|\overline{\Pi}|\psi(t)> \tag{5.15}$$

depends only on t, so that its evolution is

$$\frac{d}{dt}<\overline{\Pi}>(t) = \frac{\partial}{\partial t}<\overline{\Pi}> = <\psi(t)|\frac{1}{i\hbar}[\overline{\Pi}, H]|\psi(t)>. \tag{5.16}$$

Thus if a system has $\overline{\Pi}(H) = H^{\overline{\Pi}} = H$, then

$$\overline{\Pi} H|\psi> = \overline{\Pi}(|\psi>)\overline{\Pi}(H) = H^{\overline{\Pi}}\overline{\Pi}|\psi> = H\overline{\Pi}|\psi>, \tag{5.17}$$

so that $[\overline{\Pi}, H] = 0$ and $\overline{\Pi}$ is a constant of motion.

5.3 Real Scalar Representations

For a system of spinless particles, because $K'^{2} = 1$, one can define a real scalar[1] vector in Hilbert space as[25]

$$K'|rs> = |rs>. \tag{5.19}$$

To choose a real basis for the system, one can use the Gram-Schmidt orthogonalization process. First begin with any normalized vector $|\alpha>$, and set

$$|\alpha_{rs}> = \lambda_{\alpha}|\alpha> + \lambda_{\alpha}^{*}K'|\alpha>, \tag{5.20.1}$$

$$K'|\alpha_{rs}> = \lambda_{\alpha}|\alpha> + \lambda_{\alpha}^{*}K'|\alpha> = |\alpha_{rs}>, \tag{5.20.2}$$

---

1 Here real scalar means in the Pauli algebra an element of $\mathcal{F} = \mathcal{P}_0$.

where λ_a is to be determined by the normalization of $|\alpha_{rs}>$, and $K'|\alpha>$ should be understood by the symbolic notation

$$K'|\alpha> \equiv |\alpha^*> = <\alpha|. \tag{5.21}$$

Then one can always choose a vector $|b>$ which is orthogonal to the real scalar vector $|\alpha_{rs}>$

$$<\alpha_r|b> = 0. \tag{5.22}$$

Obviously,

$$0 \equiv <\alpha_{rs}|b>^{K'} = <K'\alpha_{rs}|K'|b> = <\alpha_{rs}|(K'|b>) \tag{5.23}$$

which means that the vector $K'|b>$ is also orthogonal to the vector $|\alpha_{rs}>$. Another real scalar vector $|b_{rs}>$

$$|b_{rs}> = \lambda_b|b> + \lambda_b^* K'|b>, \tag{5.24}$$

can be formed which is orthogonal to $|\alpha_{rs}>$. It is determined by (5.24) up to a normalization factor λ_b. Step by step, other real scalar vectors $|c_{rs}>$, $|d_{rs}>$, $\cdots$ can be formed until a complete basis is created. With this real scalar basis, all the representations can be expanded in terms of real scalar ones.

5.4 U(1) Gauge Transformation

The Hamiltonian H for a free particle without spin is

$$H = \frac{1}{2m}\vec{P}^2 = \frac{1}{2m}\vec{P}\cdot\vec{P}. \tag{5.25}$$

Thus the Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \frac{1}{2m}\left(\sum_{i=1}^{3} P_i^2\right)\psi(\vec{x},t). \tag{5.26}$$

If the wave function undergoes a local gauge transformation

$$\psi \text{ - - - } \psi' = e^{(\frac{e}{\hbar}\Lambda(\vec{x},t)}\psi, \tag{5.27}$$

where the phase factor is a pure pseudoscalar, then in order that the Schrödinger equation be invariant, H should be replaced by[1]

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})\cdot(\vec{p} - e\vec{A}) + e\phi = \frac{1}{2m}\left\{\sum_{i=1}^{3}(p_i - eA_i)^2\right\} + e\phi, \tag{5.28}$$

where ϕ and $\vec{A}$, which describe the electromagnetic field $(\vec{E}, \vec{B})$, must obey the gauge transformations:

$$\vec{A} \text{ - - - } \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$

$$\phi \text{ - - - } \phi' = \phi - \frac{\partial\Lambda}{\partial t}, \tag{5.29}$$

The Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \left\{\frac{1}{2m}(\vec{p} - e\vec{A})\cdot(\vec{p} - e\vec{A}) + e\phi\right\}\psi(\vec{x},t) \tag{5.30}$$

then describes a particle with charge e moving in the electromagnetic 4-potential $(\phi, \vec{A})$.

---

1 In this scheme, $\vec{P}$ is not an operator, only P_i is.

CHAPTER 6. SUMMARY AND DISCUSSION

In this thesis, real quantum mechanics (mainly Schrödinger theory for spinless particles) was constructed over the field of $\mathcal{F}_{\mathcal{P}_0 \oplus \mathcal{P}_3}$ in the Pauli algebra, where the $i = \sigma_1\sigma_2\sigma_3$ arises naturally as a product of the three unit basis vectors in 3-dimensional Euclidean space. In this scheme, i has geometrical meaning: it is the volume element of the algebra with all the properties of the imaginary number "i": $i^2 = -1$ and i commutes with all the other elements of $\mathcal{P}$. However, in place of complex conjugation " * ", it is the reversal " + " which changes its sign; Thus here " + " takes the role of " * ". Also, its sign changes under the spatial inversion, but not under the spatial reversal; and the phase factor associated with the $U(1)$ gauge transformation is simply the exponential of a real pseudoscalar.

In the real QM, state vectors in the Hilbert space are real scalar- plus trivector-valued in the Pauli algebra, and the hermitean conjugate is just a transpose(a reversion), so that all the observables represented by hermitean operators are symmetric. Here i itself is an anti-hermitean operator. All the other properties of Hilbert space are the same as in the usual complex QM. On the Hilbert space over the field of the real times 1 and times $i = \sigma_1\sigma_2\sigma_3$, there can be two types of antilinear transformations: one is antilinear automorphic, the other is linear anti-automorphic. Specifically the spatial inversion which changes the sign of "i" is antilinear automorphic, so that it is an anti-unitary operator. The transpose " + ", which also changes the sign of "i", is anti-automorphic, so that the spatial reversal transformation composed of spatial inversion and transpose is a linear anti-automorphism.

Under this construction, although all the numbers are real, "i" itself is a linear anti-hermitean operator in Hilbert space, and no extra anti-hermitean operator needs

to be introduced in order to express the Uncertainty Principle. The time reversal T is a combination of a transpose " + " and the unitary operator which changes the sign of i. It is an antilinear anti-automorphism, in contrast to the T in complex QM, which is an antilinear automorphism, and the interpretations of its action must be performed carefully and be based on the state vectors or an action within inner products. Specifically, when the time-reversal transformation rules for P and X were found to be the same as in classical mechanics or in complex QM, then T applied to the commutation relation of P and X, seemed to give a contradiction because of anti-automophism. However, when it acted within inner products or on the state vectors, the problem was solved.

For the parity transformation, there are two transformations to be considered: one is spatial inversion Π, the other is the spatial reversal $\overline{\Pi}$. For the former, if one wants the passive transformation and the active transformation to perform the same geometrical transformations, then the sign of "i" must change under an active transformation. It was proven that the spatial inversion is an antilinear transformation with the property of anti-unitarity. It can not be a good transformation for parity, since it gives P and X with different parities, and in different representations, they have different transformation rules. Spatial reversal, on the other hand, is an anti-automorphism; it is a linear, unitary operator as well as an hermitean operator. Since it commutes with the Hamiltonian H of a system, then it is a constant of motion. Under the spatial reversal transformation, the "i" changes its sign twice, one is related to the Π, the other is related to the transpose " + ". For calculations, in order to highlight the sign change of "i", a notation $\psi(\vec{x};i) = \psi(\vec{x})$ was used to label "i" explicitly. For applications using spatial reversal, one gets the same superselection rules for parity as in complex QM.

Using the real Pauli algebra (or multivector calculus) as a tool, this thesis has demonstrated an alternative interpretation for the "i" (quantum mechanics, electromagnetism) in complex QM as a volume element of the Pauli algebra. It was shown that there need not be any connection between i and spin. The Schrödinger theory still, as usual, describes the particle without spin. For particles with spin, the wave function should be an element of the spinor space. More work is needed to interpret quantum mechanics incorporating spin. The most significant problem is how to explain the interpretation of σ_i as both a unit vector in 3-dimensional Euclidean space and an operator on spinor space. Such a dual role for σ_i implies dual meanings for "i", even though the same tools are used to describe these two distinct spaces.

This interpretation is different from that of D. Hestenes[12b].Hestenes, reducing the Pauli equation to the Schrödinger equation, concluded that the Schrödinger equation describes a particle in an eigenstate of spin. The "i" in the usual Schrödinger equation was interpreted to be a bivector $i\sigma_3$ related to the spin. Furthermore, the spin was interpreted to be a dynamical property of a particle as opposed to a property of the intrinsic structure; the zero-point energy usually associated with the commutation relation was reinterpreted from zero-point angular momentum. According to Hestenes, there is no way to test the difference between the usual Schrödinger theory and his interpretation directly by experiment. The only direct experimental means of testing for the existence of magnetization current is by introducing a magnetic field, which necessitates the use of either the Pauli or Dirac theories. More work is needed for Hestenes's interpretation of Schrödinger theory, especially the interpretation of i associated with spin. First, if the i in QM is to be related to spin, how does one interpret complex wave functions describing particles without spin (the Klein Gordon equation and the Higgs fields) at least on purely mathematical grounds. Second, in Hestenes's approach to Schrödinger theory, if the wave function is taken to be quaternion-valued,

then how is $\vec{p}$ to be represented as a vector-valued operator? Finally, defining an angular momentum operator which includes spin proves difficult in Pauli-Schrödinger theory.

REFERENCES

[1]. S. L. Altmann, **Rotations, Quaternions, and Double Groups**, (Oxford University Press, Oxford, 1986) p.201.

[2]. F. Klein, **Elementary Mathematics From an Advanced Standpoint**, translated by E. R. Hedrick and C. A. Noble of part I of the third German edition, (Dover, New York, 1924) p.56.

[3]. P. Budinich and A. Trautman, **The Spinorial Chessboard**, ed. A. Borsellino et al, (Springer-Verlag, Berlin-Heidelberg, Germany, 1988).

[4]. C. N. Yang, " **Square Root of Minus One, Complex Phases and Erwin Schrödinger** ", Ed. C. W. Kilmister, (Cambridge University Press, 1987).

[5]. D. Hestenes and G. Sobczyk, **Clifford Algebra to Geometric Calculus**, (D. Reidel, Dordrecht 1984).

[6]. J. S. R. Chisholm and A. K. Common, **Clifford Algebras and Their Applications in Mathematical Physics.** (D. Reidel, Dordrecht, 1986).

[7]. D. Hestenes, **New Foundations for Classical Mechanics**, (D. Reidel, Dordrecht, 1987); **Spacetime Algebra**, (Gordon and Breach, New York, 1966).

[8]. W. E. Baylis and John Huschilt, **Classical Mechanics — a Vector-algebra Approach**, (Windsor, U. Of Windsor, 1989).

[9]. B. Jancewicz, **Multivectors and Clifford Algebra in Electrodynamics**, (World Scientific Pub., Singapore, 1989).

[10]. I. M. Benn and R. W. Tucker, **An Introduction to Spinors and Geometry with Applications in Physics**, (Adam Hilger, Bristol, 1987).

[11]. N. Salingaros, Found. Of Phys. **14** (1984) 777; J. Math. Phys. **25** (1984) 706; Phys. Rev. D 31 (1985) 3150.

[12] a): D. Hestenes, J. Math. Phys. **8** (1967) 798; **14** (1973) 893; **16** (1975) 556; Found. Of Phys. **12** (1981) 153; " **Clifford Algebra and the Interpretation of Quantum Mechanics** " **in Clifford Algebras and Their Applications in Mathematical Physics**, eds. J. S. R. Chisholm and A. K. Common, (D. Reidel, Dordrecht, 1986), pp.321-346. b): R. Gurtler and D. Hestenes, J. Math. Phys. 16 (1975) 573; D. Hestenes, Am. J. Phys. 47 (1979) 399.

[13] H. T. Cho, Adel Diek and R. Kantowski, J. Math. Phys. **31** (1990) 2192.

[14] W. E. Baylis, Am. J. Phys. **48** (1980) 918; W. E. Baylis and George Jones, J. Phys. A: Math. Gen. **22** (1989) 1; 17.

[15] W. E. Baylis, **Classical Eigenspinors and the Dirac Equation**. Submitted for publication.

[16] D. Kosokowsky, Work in progress on " Pauli algebra approach to Weak and E&M theory " at University of Windsor.

[17] G. Jones, Work in progress on " Pauli algebra approach to general relativity " at University of Windsor.

[18] J.V.Neumann and G.D.Birkhoff, Annals of Math. **37** (1936) 823.

[19] E. C. G. Stueckelberg, Helv. Phys. Acta. 33 (1960) 727; E. C. G. Stueckelberg and M. Guenin, Helv. Phys. Acta. **34** (1961) 621;D. Finkelstein, J. M. Jauch and D. Speiser, J. Math. Phys. **3** (1962) 207; **4** (1963) 136; 788.

[20] S. L. Adler, Phys. Rev. Lett, **57** (1986) 167; Phys. Rev. **D** 34 (1986) 1871.

[21] S. Maclane and G. Birkhoff, **Algebra**, (The Macmillan Company, New York, 1967); **J. B. Fraleigh, A First Course in Abstract Algebra**, (Addison-Wesley Publishing Company, New York, 1989).

[22] E. Prugovecki, **Quantum Mechanics in Hilbert Space** (Academic Press, New York, 1981).

[23] J. Von Neumann, **Mathematical Foundations of Quantum Mechanics**, (Princeton University Press, Princeton, 1955).

[24] E. P Wigner, **Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra**, translated by J. J. Griffin, (Academic Press Inc. New York, 1959); R. G. Sachs, **The Physics of Time Reversal**, (University of Chicago Press, Chicago, 1987).

[25] C. Cohen-Tannoudji, B. Diu and F. Laloë, **Quantum Mechanics**, Vol.I, translated by. S. R. Hemley, N. Ostrowsky and D. Ostrowsky, (Hermann, Paris, 1977).

[26] A. Messiah, **Quantum Mechanics**, Vols. I-II, translated by G. M. Temmer, (North-Holland, Amsterdam 1961).

[27] P. A. M. Dirac, **The Principle of Quantum Mechanics**, (Oxford, London, 1958).

VITA AUCTORIS

NAME:	Jiansu WEI
PLACE OF BIRTH:	Sichuan, P. R. China
YEAR OF BIRTH:	1957
EDUCATION:	Chongqing Teacher's College, Chongqing, P. R. China, 1978-1982 B.Sc
	Sichuan University, Chengdu, P. R. China, 1983-1986 M.Sc
	University of Windsor, Windsor, Ontario, 1989-1991 M.Sc